WEEK 4 NOTES

1. CONTINUATION OF WAVE EQUATION ON THE REAL LINE

We continue the theory wave equation

$$U_{tt} = c^2 U_{xx}, \qquad c \equiv \sqrt{\frac{F}{\rho}}. \tag{1.1}$$

Recall that by the following change of variables

$$u = x - ct, \qquad v = x + ct. \tag{1.2}$$

the wave equations becomes

$$U_{uv} = 0$$

and this gives the general solutions

$$U(x,t) = F(x+ct) + G(x-ct).$$

1.1. **Where does the change of variables come from?** To explain the change of variables, one observes that the wave equation can be rewritten as

$$U_{tt} - c^2 U_{xx} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) U = 0.$$

Letting

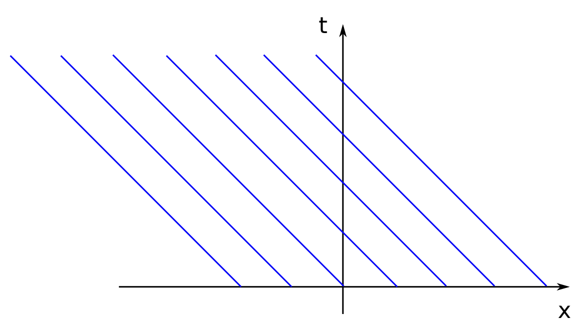
$$W \equiv \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) U,$$

then

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) W = \frac{\partial W}{\partial t} - c\frac{\partial W}{\partial x} = W_t - cW_x = 0.$$

Thus, W satisfies a first order pde with constant coefficients —we have already studied the solutions to this equation. The characteristics are lines with negative slope $dt/dx = -1/c$ (negative slope) so that

$$x + ct = \text{constant.}$$

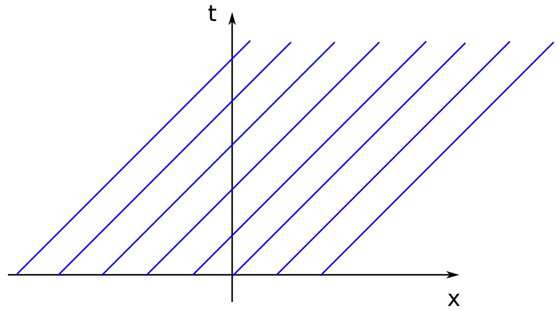


Once we know W one has to solve the equation

$$U_t + cU_x = W$$

which is, again, a first order pde with constant coefficients —observe, however, that the equation is inhomogeneous. The slope of the characteristics is $dt/dx = 1/c$ (positive slope) so that

$$x - ct = \text{constant}.$$



Note. *Thus, the wave equation has two sets of characteristics —that is, there is information travelling in two directions: to the left and to the right.*

2. Some invariant properties of wave equations on the real line

From a given solution $U(x,t)$ to the wave equation (1.1) on the real line $x \in \mathbb{R}$, we can construct new solutions to the equation

Proposition 2.1. *If $U(x,t)$ is a solution to the wave equation* (1.1) *on the real line, so are*

$$V(x,t) = U(\alpha x, \alpha t), \textit{for any } \alpha \in \mathbb{R},$$
$$W(x,t) = U(x, -t).$$

Proposition 2.2. *Let $U_1(x,t)$ solves the the advection equation $(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x})U_1 = 0$ on the real line,*
and $U_2(x,t)$ solves the the advection equation $(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x})U_2 = 0$ on the real line.
Then both U_1 and U_2 solves the wave equation (1.1) *on the real line.*

We will leave it as an exercise to show these 2 propositions. Problem sets also contains some similar questions about the invariant properties.

3. Conservation of energy

Consider the wave equation on the line:

$$U_{tt} = c^2 U_{xx}, \qquad x \in \mathbb{R}$$
$$U(x,0) = f(x), \qquad U_t(x,0) = g(x),$$

where $f(x),\ g(x) = 0$ for $|x| > R$, with R some big number. This means that $f(x)$ and $g(x)$ vanish for large $|x|$ —functions of this type are said to have *compact support*.

3.1. **Derivation.** Multiply now the wave equation by U_t on both sides:

$$U_t U_{tt} = c^2 U_{xx} U_t.$$

Observing that

$$U_t U_{tt} = \frac{1}{2}\frac{\partial}{\partial t}\left(U_t^2\right),$$

one has then that

$$\frac{1}{2}\frac{\partial}{\partial t}\left(U_t^2\right) - c^2 U_{xx} U_t = 0.$$

Integrating over the real line one then gets that

$$\begin{aligned}
0 &= \int_{\infty}^{-\infty}\left(\frac{1}{2}\frac{\partial}{\partial t}\left(U_t^2\right) - c^2 U_{xx}U_t\right)dx \\
&= \frac{d}{dt}\int_{-\infty}^{\infty}\frac{1}{2}U_t^2 dx - c^2\int_{-\infty}^{\infty}U_{xx}U_t dx \\
&= \frac{d}{dt}\left(\int_{-\infty}^{\infty}\frac{1}{2}U_t^2 dx\right) - \left(U_t U_x\Big|_{-\infty}^{\infty} - c^2\int_{-\infty}^{\infty}U_x U_{xt} dx\right) \\
&= \frac{d}{dt}\int_{-\infty}^{\infty}\frac{1}{2}U_t^2 dx + c^2\int_{-\infty}^{\infty}U_x U_{xt} dx,
\end{aligned}$$

where to pass from the second to the third line we have used integration by parts and in the third line that $U(x,t) = 0$ if $|x| \to \infty$. Finally, observing that

$$U_x U_{xt} = \frac{1}{2}\frac{\partial}{\partial t}(U_x^2),$$

one concludes that

$$0 = \frac{d}{dt}\int_{-\infty}^{\infty}\frac{1}{2}U_t^2 dx + c^2\int_{-\infty}^{\infty}\frac{1}{2}\frac{\partial}{\partial t}(U_x^2)dx,$$

so that

$$\frac{d}{dt}\left(\frac{1}{2}\int_{-\infty}^{\infty}\left(U_t^2 + c^2 U_x^2\right)dx\right) = 0.$$

In other words, the quantity in brackets is constant in time. This calculation suggests the following definition:

Definition 3.1. The **energy** $E[U](t)$ of a solution to the wave equation is given by

$$E[U](t) \equiv \frac{1}{2}\int_{-\infty}^{\infty}\left(U_t^2 + c^2 U_x^2\right)dx.$$

Hence, the previous calculations show that

$$\frac{d}{dt}E[U](t) = 0,$$

that is, the energy is conserved —i.e. independent of t (**law of conservation of total energy**). The term $\int U_t^2/2$ is called the **kinetic energy** and $\int \frac{c^2}{2}U_x^2$ the **potential energy**.

3.2. **An application: uniqueness of solutions.** In this subsection we show how the total energy can be used to show that a solution to the initial value problem

$$U_{tt} - c^2 U_{xx} = 0, \qquad x \in \mathbb{R}$$
$$U(x,0) = f(x), \qquad U_t(x,0) = g(x)$$

if it exists, then it must be unique.

Suppose one has 2 solutions U_1 and U_2 an let $W \equiv U_1 - U_2$. As the wave equation is linear one has that

$$W_{tt} - c^2 W_{xx} = 0,$$
$$W(x,0) = 0, \qquad W_t(x,0) = 0.$$

The energy of W can be directly computed to be

$$\begin{aligned} E[W](t) &= E[W](0) \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \big(W_t^2(x,0) + c^2 W_x^2(x,0)\big)dx, \\ &= 0. \end{aligned}$$

This means, in particular, that

$$\int_{-\infty}^{\infty} \big(W_t^2(x,t) + c^2 W_x^2(x,t)\big)dx = 0,$$

but $W_t^2 \geq 0$, $W_x^2 \geq 0$ so that, in order for the integral to vanish one actually needs

$$W_t(x,t) = 0, \qquad W_x(x,t) = 0.$$

Thus $W(x,t)$ is constant for all x, t. But $W(x,0) = 0$ so that $W(x,t) = 0$. Hence, $U_1 = U_2$ —that is, the solution is unique.

4. More examples for hyperbolic PDEs on the real line

Example 4.1. The Gourset problem has 2 mixed boundary conditions. Let's consider a case with wave speed $c = 1$.

$$U_{tt} = u_{xx}, x \in \mathbb{R}$$
$$U|_{x-t=0} = x, U|_{x+t=0} = x^2.$$

Recall the general solution is given by

$$U(x,t) = F(x+t) + G(x-t).$$

When $x - t = 0$, we have $x = t$, so

$$x = U|_{x-t=0} = F(x+x) + G(x-x) = F(2x) + 0.$$

Thus $F(x) = \frac{x}{2} - G(0)$.

Moreover, plugging in $x = 0$ gives $F(0) + G(0) = 0$.

Similarly, using the $x + t = 0$ condition (giving x=-t), we get

$$x^2 = U|_{x+t=0} = F(0) + G(2x).$$

Thus $G(x) = \frac{x^2}{4} - F(0)$.

So now the solution is then

$$U(x,t) = \frac{x+t}{2} - G(0) + \frac{(x-t)^2}{4} - F(0) = \frac{x+t}{2} + \frac{(x-t)^2}{4},$$

using that $F(0) + G(0) = 0$.

Example 4.2. The following equation is hyperbolic.

$$U_{xx} - U_{xt} - 2U_{tt} = 0.$$

To find the general solutions, we use the argument from 1.2 in Week 4 notes by factoring the 2nd oder PDE to two 1st order PDEs.

We get the equation

$$0 = U_{xx} - U_{xt} - 2U_{tt} = (\frac{\partial}{\partial x} + \frac{\partial}{\partial t})(\frac{\partial}{\partial x} - 2\frac{\partial}{\partial t})U.$$

This is because

$$\begin{aligned}&(\frac{\partial}{\partial x} + \frac{\partial}{\partial t})(\frac{\partial}{\partial x} - 2\frac{\partial}{\partial t})U\\ =&(\frac{\partial}{\partial x} + \frac{\partial}{\partial t})(U_x - 2U_t)\\ =&U_{xx} + U_{xt} - 2U_{xt} - 2U_{tt}\\ =&U_{xx} - U_{xt} - 2U_{tt}.\end{aligned}$$

Now, we denote by $W = (\frac{\partial}{\partial x} - 2\frac{\partial}{\partial t})U$, then the equation becomes 2 first order PDEs

$$\begin{aligned}&(\frac{\partial}{\partial x} + \frac{\partial}{\partial t})W = 0\\ &(\frac{\partial}{\partial x} - 2\frac{\partial}{\partial t})U = W\end{aligned}$$

Solve the first equation for W as a first order linear homogeneous PDE, we get

$$W(x, t) = f(x - t).$$

The second equation now is an inhomogeneous first order linear PDE

$$U_x - 2U_t = f(x - t).$$

The characteristic lines are

$$2x + t = C,$$

and along the characteristic lines, $t = C - 2x$.

So the PDE becomes ODE along the characteristsic lines.

$$\frac{d}{dx}U(x, t(x)) = \frac{U_x - 2U_t}{1} = f(x - t) = f(x - (C - 2x)) = f(3x - C).$$

Integrate both sides with respect to x, get

$$U(x, t) = \int f(3x - C)dx = F(3x - C) + G(C),$$

for any F, G.

Plugging in the relation $C = 2x + t$ from the characteristic equation, we get the general solution

$$U(x, t) = F(3x - 2x - t) + G(2x + t) = F(x - t) + G(2x + t).$$

If there is imposed initial conditions (initial position and initial velocity), you can then use them to determine F and G as we did before.

5. THE WAVE EQUATION ON THE HALF-LINE: REFLECTION

In this section we analyse with more detail the phenomenon of reflection of waves on a wall. This problem naturally leads one to consider solutions of the wave equation on the half-line.The latter is modelled by the problem

$$U_{tt} - c^2 U_{xx} = 0,$$

with boundary condition

$$U(0,t) = 0, \qquad t > 0$$

and initial conditions

$$\begin{aligned} &U(x,0) = f(x), \\ &U_t(x,0) = g(x), \qquad x \geq 0. \end{aligned}$$

Boundary conditions like $U(0,t) = 0$ specifying the value of the solution on some boundary (in this case the t-axis) are known as **Dirichlet boundary conditions**. In this case it models a solid wall through which the wave cannot propagate.

To construct solutions to the above problem we will make use of D'Alembert's formula for solutions of the wave equation on the *whole* real line. Notice, however, that the initial conditions described by the functions f and g are only given on the half-line —i.e. for $x \geq 0$. To get around this problem we consider *odd extensions* of the functions f and g. More precisely, we define

$$F(x) \equiv \begin{cases} f(x) & x \geq 0 \\ -f(-x) & x < 0 \end{cases}$$

and

$$G(x) \equiv \begin{cases} g(x) & x \geq 0 \\ -g(-x) & x < 0 \end{cases}$$

—see the figure below for a depiction of the idea behind an odd extension of a function.

Now, with the help of the functions F and G we consider the problem on the the whole real line given by

$$\begin{aligned} &V_{tt} - c^2 V_{xx} = 0, \\ &V(x,0) = F(x), \\ &V_t(x,0) = G(x), \qquad x \in \mathbb{R}. \end{aligned}$$

The solution to the above problem is given then by D'Alembert's formula as

$$V(x,t) = \frac{1}{2}\left(F(x+ct) + F(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)ds. \tag{5.1}$$

To see what the relation between the above solution and the problem on the half-line is evaluate $V(x,t)$ as given above on the t-axis —i.e. at $x = 0$. One has that

$$\begin{aligned} V(0,t) &= \frac{1}{2}\left(F(ct) + F(-ct)\right) + \frac{1}{2c}\int_{-ct}^{ct} G(s)ds \\ &= 0, \end{aligned}$$

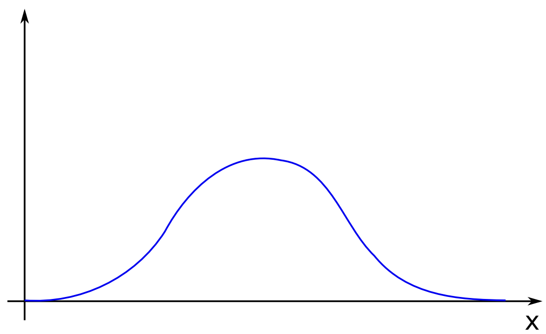
where it has been used that $F(-x) = -F(x)$ (because of the odd extension) and the fact that the integral of an odd function on a symmetric interval must vanish.

Accordingly $V(x,t)$ satisfies the boundary conditions for the problem on the half-line. Moreover, $V(x,t)$ satisfies the wave equation and the initial conditions for V and U coincide for $x \geq 0$. Thus, one has that

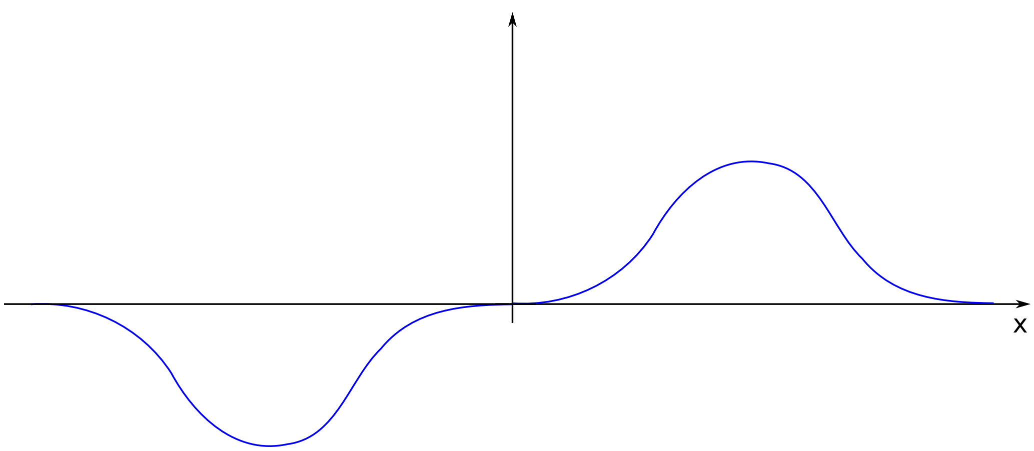
$$U(x,t) = V(x,t), \qquad x \geq 0.$$

This last result make use of the **uniqueness** of solutions to the wave equation —a topic not yet covered in the course!

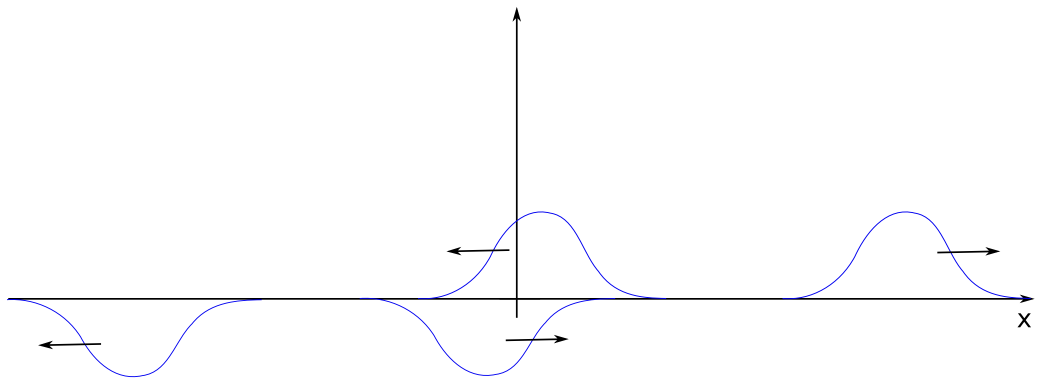
The claim is that formula (5.1) describes the phenomenon of reflection of waves. To see this better consider a situation for which $g(x) = 0$ and $f(x)$ has the form of a bump:



The odd extension of the bump has the form



As we have seen in the discussion of the interpretation of D'Alembert's formula, initial profiles split into two smaller profiles, half the size. In our particular case, each bump will split into two bumps half the size, one travelling to the right, the other to the left. The negative bump travelling to the right describes the reflected wave! When thinking about this problem we only need to concentrate on the solution for $x \geq 0$ and ignore the solution for $x < 0$.



6. Waves on an interval

In this section we will study the vibrations of a finite string with fixed ends. This problem will help to illustrate an important method to solve linear pde's —the so-called *method of separation of variables*. This method applies, more generally, to linear homogeneous pde's with homogeneous boundary conditions. The key to this method is the *principle of superposition*.

Assume that the string has ends given by $x = 0$ and $x = L$. The problem we want to study can be formulated mathematically as finding solutions to

$$U_{tt} - c^2 U_{xx} = 0 \tag{6.1}$$

with *boundary conditions* given by

$$U(0,t) = 0, \qquad U(L,t) = 0, \tag{6.2}$$

and *initial conditions* given by

$$U(x,0) = f(x), \qquad U_t(x,0) = g(x).$$

6.1. **The method of separation of variables.** In the method of separation of variables one looks for solutions to the wave equation (6.1) of the form

$$U(x,t) = X(x)T(t) \tag{6.3}$$

where X is a function of x and T of t only. Substitution of (6.3) into the wave equation (6.1) gives

$$X\ddot{T} = c^2 X'' T$$

where $\dot{}$ denotes differentiation with respect to t and $'$ differentiation with respect to x. Dividing by c^2XT one finds that

$$\frac{1}{c^2}\frac{\ddot{T}}{T} = \frac{X''}{X}. \tag{6.4}$$

The key observation in the method of separation of variables is that the left hand side of equation (6.4) only depends on t while the right hand side only x. If these to sides are to be equal it means that they have to be both constant. Let us denote this constant by $-\lambda$ —the minus sign is conventional. The constant λ is called the *separation constant*.

Thus one has that

$$\frac{1}{c^2}\frac{\ddot{T}}{T} = \frac{X''}{X} = -\lambda.$$

From the latter one obtains two ordinary differential equations —one for $T(t)$ and one for $X(x)$. Namely, one has that

$$\ddot{T} = -c^2\lambda T, \tag{6.5a}$$

$$X'' = -\lambda X. \tag{6.5b}$$

6.1.1. *The eigenvalue problem.* We begin by looking at equation (6.5b). The boundary conditions (6.2) imply that

$$X(0) = 0, \qquad X(L) = 0.$$

The combination

$$X'' = -\lambda X, \tag{6.6a}$$

$$X(0) = 0, \qquad X(L) = 0, \tag{6.6b}$$

is known as the *eigenvalue problem*. To solve it one needs to find all non-trivial solutions to (6.6a)-(6.6b). Observe that trivially $X(x) = 0$ is a solution.

Note. *Eigenvalue problems arise in the context of Linear Algebra. In that subject matrices are linear operators. In Differential Equations the operator is the action of taking two derivatives of the function X.*

An important property of the eigenvalue problem (6.6a)-(6.6b) is that $\lambda > 0$. To see this, rewrite (6.6a) as

$$X'' + \lambda X = 0.$$

Multiplying by X and integrating one obtains

$$\int_0^L \big(XX'' + \lambda X^2\big)dx = XX'\Big|_0^L - \int_0^L (X')^2 dx + \lambda\int_0^L X^2 dx = 0.$$

Where we have used integration by parts and the boundary conditions (6.6b) to eliminate the term $XX'|_0^L$. Hence,

$$0 < \int_0^L (X')^2 dx = \lambda \int_0^L X^2 dx.$$

The latter is only possible if $\lambda > 0$.

Given that $\lambda > 0$, the general solution to equation (6.6a) is given by

$$X(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x), \qquad c_1,\ c_2 \quad \text{constants.}$$

Now, using the boundary conditions one has

$$X(0) = c_1 = 0,$$
$$X(L) = c_2\sin(\sqrt{\lambda}L) = 0.$$

So, either $c_2 = 0$ which forces $X(x) = 0$ or $\sqrt{\lambda}L = n\pi$ with $n = 1, 2, 3, \ldots$. Accordingly, one defines the eigenvalues

$$\lambda_n \equiv \left(\frac{n\pi}{L}\right)^2, \qquad n = 1, 2, 3, \ldots.$$

The corresponding eigenfunction is given by

$$X_n(x) \equiv \sin\left(\frac{n\pi x}{L}\right).$$

6.1.2. *The equation for $T(t)$.* Now, we use the information from the eigenvalue problem to solve the equation

$$\ddot{T} = -c^2\lambda T.$$

As $\lambda > 0$ the general solution is given by

$$T(t) = d_1\cos(\sqrt{\lambda}ct) + d_2\sin(\sqrt{\lambda}ct).$$

6.1.3. *The general solution.* Combining the expressions obtained in the previous paragraphs, for a given n the solution consists of products

$$U_n(x,t) = a_n\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right)$$

with constants a_n, b_n. Now, recalling that the wave equation is linear, one has that the *principle of superposition* applies: the sum of two solutions is also a solution. Taking this to the extreme one has that

$$\begin{aligned} U(x,t) &= \sum_{n=1}^{\infty} U_n(x,t) \\ &= \sum_{n=1}^{\infty}\left[a_n\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right)\right] \end{aligned} \tag{6.7}$$

Next, to determine the coefficients a_n, b_n, we will need to make use of the initial conditions $U(x,0) = f(x)$ and $U_t(x,0) = g(x)$.