# WEEK 3 NOTES

## 1. SECOND ODER PARTIAL DIFFERENTIAL EQUATIONS

In this section we briefly look at the classification of second order partial differential equations with constant coefficients.

### 1.1. **Introduction.**

The most general second order partial differential equation with constant coefficients is given by

$$aU_{xx} + 2bU_{xy} + cU_{yy} + dU_x + eU_y + fU = h(x,y) \tag{1.1}$$

with

$$a, \quad b, \quad c, \quad d, \quad e, \quad f,$$

are constants and $h(x,y)$ is an arbitrary function. The terms with the highest order derivatives, namely

$$aU_{xx} + 2bU_{xy} + cU_{yy} \tag{1.2}$$

are called the *principal part*. It determines the character of the solutions of the equation. In the following, to avoid messy computations, we consider only the principal part —i.e. we set $d$, $e$ and $f$ to zero. Particular cases of equation (1.1) are

$$\begin{aligned} &U_{xx} - U_{tt} = 0 \qquad \text{(wave equation)},\\ &U_{xx} + U_{yy} = 0 \qquad \text{(Laplace equation)},\\ &U_{xx} - U_t = 0 \qquad \text{(heat equation)}. \end{aligned}$$

The solutions to each of these equations have a completely different behaviour. In the following, we will see that, in a sense, these are the only possibilities.

### 1.2. **Quadratic forms.**

The basic observation is the following: compare the principal part (1.2) with the *quadratic form*

$$ax^2 + 2bxy + cy^2.$$

We know from basic geometry that the solutions to the equation defined by this quadratic form represents a conic section —i.e. a hyperbola, a parabola or an ellipse. The type of conic section depends on the coefficients in the quadratic form. More precisely, completing squares one has that

$$ax^2 + 2bxy + cy^2 = a\left(\left(x + \frac{b}{a}y\right)^2 + \left(\frac{ac-b^2}{a^2}\right)y^2\right).$$

One then has the following classification:

$$\begin{aligned} b^2 - ac &> 0 \qquad \text{hyperbola},\\ b^2 - ac &= 0 \qquad \text{parabola},\\ b^2 - ac &< 0 \qquad \text{ellipse}. \end{aligned}$$

One can do something similar with the principal part (1.2). One can readily check that

$$aU_{xx} + 2bU_{xy} + cU_{yy} = a\left(\left(\frac{\partial}{\partial x} + \frac{b}{a}\frac{\partial}{\partial y}\right)^2 + \left(\frac{ac-b^2}{a^2}\right)\frac{\partial}{\partial y^2}\right)U.$$

Accordingly, one classifies the pde's according to the same criteria as for the quadratic forms —more precisely, one says that (1.1) is

$$\begin{aligned} b^2 - ac > 0 &\qquad \text{hyperbolic pde,} \\ b^2 - ac = 0 &\qquad \text{parabolic pde,} \\ b^2 - ac < 0 &\qquad \text{elliptic pde.} \end{aligned}$$

One can readily check that

| | |
|---|---|
| wave equation | hyperbolic, |
| Laplace equation | elliptic, |
| heat equation | parabolic. |

1.3. **A change of variables.** Consider now new coordinates $(x', y')$ given by

$$\begin{aligned} x' &= x, \\ y' &= -\frac{b}{a}x + y, \end{aligned}$$

so that

$$\begin{aligned} x &= x', \\ y &= y' + \frac{b}{a}x. \end{aligned}$$

Using the chain rule for partial derivatives one finds that

$$\begin{aligned} \frac{\partial}{\partial y'} &= \frac{\partial}{\partial y}, \\ \frac{\partial}{\partial x'} &= \frac{\partial}{\partial x} + \frac{b}{a}\frac{\partial}{\partial y}. \end{aligned}$$

Substituting the above into the principal part (1.2) a calculation readily gives

$$aU_{xx} + 2bU_{xy} + cU_{yy} = a\left(U_{x'x'} + \left(\frac{ac-b^2}{a^2}\right)U_{y'y'}\right).$$

This is the *canonical form* of the *principal part*. And we see that the principal part can always be changed to this canonical form, without mixed derivatives $U_{xy}$!

Now, if $ac - b^2 < 0$ one can write

$$\begin{aligned} U_{x'x'} + \left(\frac{ac-b^2}{a^2}\right)U_{y'y'} &= U_{x'x'} - \frac{|ac-b^2|}{|a|^2}U_{y'y'} \\ &= \left(\frac{\partial}{\partial x'} + \frac{\sqrt{|ac-b^2|}}{|a|}\frac{\partial}{\partial y'}\right)\left(\frac{\partial}{\partial x'} - \frac{\sqrt{|ac-b^2|}}{|a|}\frac{\partial}{\partial y'}\right)U. \end{aligned}$$

In fact, one can eliminate the factor $\sqrt{|ac-b^2|}/|a|$ by a further change of variables.

**Note.** *The classification also works if the coefficients depend on the coordinates. In that case the character of the equation can change from point to point. As an example one has the equation*

$$U_{xx} + xU_{yy} = 0.$$

We will now focus on hyperbolic equations in the coming weeks. A typical example is the wave equations

## 2. THE WAVE EQUATION IN $1+1$ DIMENSION

The wave equation in $1+1$ dimension is

$$U_{tt} - c^2 U_{xx} = 0 \tag{2.1}$$

with $c$ a constant (**wave speed**) and $x \in I \subseteq \mathbb{R}$, $t > 0$ —i.e. $I$ is an interval which can be finite, semi-infinite or infinite. The equation is supplemented by *initial conditions*

$$U(x,0) = f(x), \qquad U_t(x,0) = g(x),$$

and, possibly, also *boundary conditions* if $I \neq \mathbb{R}$.

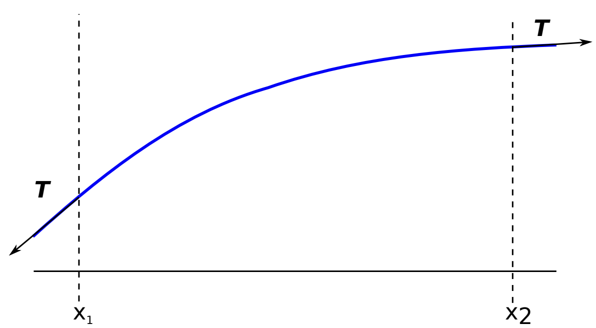
**Note.** *In $3+1$ dimensions the wave equation takes the form*

$$U_{tt} - c^2(U_{xx} + U_{yy} + U_{zz}) = 0.$$

The wave equation arises in problems describing the vibration of strings and membranes. More generally, the equations describe sound waves, electromagnetic waves, seismic waves, gravitational waves, propagation of epidemics, movement of populations, ...

### 2.1. **The vibrating string.**

Consider, in the following, a *flexible, elastic, homogeneous string* of length $L$ undergoing small transverse vibrations. Assume that the motion is restricted to a plane, and let $U(x,t)$ be the displacement from equilibrium position at time $t$ and position $x$.




If the string is perfectly flexible, then the force (*tension*) responsible for the displacement is directed tangentially along the string and is constant in time since the string is homogeneous. The position of the string at a point $x$ is then given by $\big(x, U(x,t)\big)$ and the slope of the string at $x$ is that of the tangent. The tangent vector at a point $x$ is given by

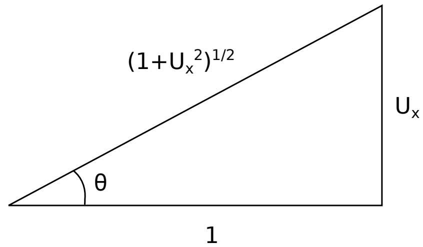
$$\frac{d}{dx}\big(x, U(x,t)\big) = \big(1, U_x(x,t)\big).$$

The key to obtaining an equation for $U(x,t)$ is **Newton's second law**

$$\vec{F} = m\vec{a}.$$

Now, from the diagram one has that

$$\cos\theta = \frac{1}{\sqrt{1+U_x^2}}, \qquad \sin\theta = \frac{U_x}{\sqrt{1+U_x^2}}.$$




The tension $\vec{F}$ is then given by

$$\vec{F} = F(x)\big(\cos\theta, \sin\theta\big) = \frac{F}{\sqrt{1+U_x^2}}\big(1, U_x\big),$$

where $F = F(x)$ is the norm of $\vec{F}$ and is assumed to be independent of time —see above. From the discussion above we have that the position of an element of string is then given by

$$\vec{x} = \big(x, U(x,t)\big)$$

so that its velocity and acceleration are given, respectively, by

$$\dot{\vec{x}} = \big(0, U_t\big), \qquad \ddot{\vec{x}} = \big(0, U_{tt}\big),$$

where the overdot $\dot{}$ denotes differentiation with respect to $t$. We can now compute the force along a segment of string $[x_1, x_2]$ using Newton's law:

$$\frac{F(x)}{\sqrt{1+U_x^2}}\big(1, U_x\big)\bigg|_{x_1}^{x_2} = \int_{x_1}^{x_2} \rho \cdot \big(0, U_{tt}(s,t)\big)\mathrm{d}s,$$

where $\rho$ is the density of the string (mass/unit length) which we assume to be constant. As this is a vector expression it implies two equations for the $x$ and $y$ components. The $x$ component gives the equation

$$\frac{F(x)}{\sqrt{1+U_x^2}}\bigg|_{x_1}^{x_2} = 0, \tag{2.2}$$

while the $y$ component gives

$$\frac{F(x)}{\sqrt{1+U_x^2}}U_x\bigg|_{x_1}^{x_2} = \int_{x_1}^{x_2} \rho U_{tt}(s,t)ds. \tag{2.3}$$

Now, as $U_x$ is assumed to be small, then using Taylor series one has that

$$\sqrt{1+U_x^2} \approx 1 + \frac{1}{2}U_x^2 + \cdots \approx 1.$$

Using this approximation it follows from (2.2) that $F(x)$ is constant —i.e. independent of $x$. Equation (2.3) then gives

$$F\Big(U_x(x_2,t) - U_x(x_1,t)\Big) = \int_{x_1}^{x_2} \rho U_{tt}(s,t)ds.$$

Now, the fundamental theorem of calculus then gives that

$$F\big(U_x(x_2,t) - U_x(x_1,t)\big) = F\int_{x_1}^{x_2} U_{xx}(s,t)ds.$$

Hence,

$$F\int_{x_1}^{x_2} U_{xx}(x,t)ds = \int_{x_1}^{x_2} \rho U_{tt}(s,t)ds.$$

As the points $x_1$ and $x_2$ are arbitrary the integrands must be equal so that

$$\frac{F}{\rho}U_{xx} = U_{tt}.$$

We write the latter as

$$U_{tt} = c^2 U_{xx}, \qquad c \equiv \sqrt{\frac{F}{\rho}}. \tag{2.4}$$

This is the (homogeneous) *wave equation*. The constant $c$ is called the **wave speed**.

2.2. **Computing the general solution.** Consider the change of variables

$$u = x - ct, \qquad v = x + ct. \tag{2.5}$$

Using the chain rule one has that

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v},\\ \frac{\partial}{\partial t} &= \frac{\partial u}{\partial t}\frac{\partial}{\partial u} + \frac{\partial v}{\partial t}\frac{\partial}{\partial v} = c\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial u}\right).\end{aligned}$$

The second derivatives are computed as

$$\begin{aligned}\frac{\partial^2}{\partial x^2} &= \left(\frac{\partial}{\partial x}\right)^2 = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\\ &= \frac{\partial^2}{\partial u^2} + 2\frac{\partial^2}{\partial u \partial v} + \frac{\partial^2}{\partial v^2},\\ \frac{\partial^2}{\partial t^2} &= \left(\frac{\partial}{\partial t}\right)^2 = c^2\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial u}\right)\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial u}\right)\\ &= c^2\left(\frac{\partial^2}{\partial v^2} - 2\frac{\partial^2}{\partial u \partial v} + \frac{\partial^2}{\partial u^2}\right).\end{aligned}$$

Thus, one has that

$$\begin{aligned}U_{tt} - c^2 U_{xx} &= c^2\left(\frac{\partial^2}{\partial v^2} - 2\frac{\partial^2}{\partial u \partial v} + \frac{\partial^2}{\partial u^2}\right)U - c^2\left(\frac{\partial^2}{\partial u^2} + 2\frac{\partial^2}{\partial u \partial v} + \frac{\partial^2}{\partial v^2}\right)U\\ &= -4c^2\frac{\partial^2 U}{\partial u \partial v}.\end{aligned}$$

Hence, we have transformed the original wave equation (3.1) into

$$\frac{\partial^2 U}{\partial u \partial v} = 0. \tag{2.6}$$

To solve equation (2.6) we notice that

$$\frac{\partial^2 U}{\partial u \partial v} = \frac{\partial}{\partial u}\left(\frac{\partial U}{\partial v}\right),$$

so that integrating with respect to $u$ one has

$$\int \frac{\partial}{\partial u}\left(\frac{\partial U}{\partial v}\right)du = \frac{\partial U}{\partial v} = f(v),$$

with $f(v)$ an arbitrary function of $v$. Integrating now with respect to $v$ one gets

$$U(u,v) = \int f(v)dv + G(u),$$

with $G(u)$ an arbitrary function of $u$. Now, observe that the integral $\int f(v)dv$ is an arbitrary function of $v$ so that one can write

$$U(u,v) = F(v) + G(u)$$

where $F(v)$ is another arbitrary function of $v$. Writing the latter in terms of the coordinates $(x,y)$ one finds that

$$U(x,t) = G(x-ct) + F(x+ct). \tag{2.7}$$

This is the general solution to the wave equation (3.1).

## 3. The wave equation on the real line

We continue the theory wave equation

$$U_{tt} = c^2 U_{xx}, \qquad c \equiv \sqrt{\frac{F}{\rho}}. \tag{3.1}$$

on the real line so that there are no boundary conditions —physically, this means that we consider an infinitely long vibrating string. This is a useful *idealisation*.

### 3.1. **Solution in terms of initial conditions.**

Now, suppose one has the initial conditions

$$U(x,0) = f(x), \qquad U_t(x,0) = g(x).$$

One needs to two initial conditions as the equation is second order. At $t = 0$ the general solution deduced last week gives

$$U(x,0) = G(x) + F(x) = f(x). \tag{3.2}$$

Moreover, a direct computation using the chain rule gives

$$U_t(x,t) = cF'(x+ct) - cG'(x-ct),$$

so that

$$U_t(x,0) = cF'(x) - cG'(x) = g(x) \tag{3.3}$$

Differentiating (3.2) with respect to $x$ one obtains the system of equations

$$\begin{aligned} f'(x) &= G'(x) + F'(x), \\ g(x) &= cF'(x) - cG'(x). \end{aligned}$$

Adding and subtracting these equations one finds that

$$\begin{aligned} F'(x) &= \frac{1}{2c}\big(g(x) + cf'(x)\big), \\ G'(x) &= \frac{1}{2c}\big(cf'(x) - g(x)\big). \end{aligned}$$

Integrating the first of these equations with respect to $x$ one finds that

$$\begin{aligned} F(x) - F(0) &= \int_0^x \frac{1}{2c}\big(g(s) + cf'(s)\big)ds \\ &= \frac{1}{2}\big(f(x) - f(0)\big) + \frac{1}{2c}\int_0^x g(s)ds, \end{aligned}$$

where in the second line we have used the *Fundamental theorem of Calculus*. Moreover, using this last expression one has that

$$
\begin{aligned}
G(x) &= f(x) - F(x) \\
&= f(x) - \frac{1}{2}f(x) + \frac{1}{2}f(0) - \frac{1}{2c}\int_0^x g(s)ds - F(0) \\
&= \frac{1}{2}f(x) + \frac{1}{2}f(0) - \frac{1}{2c}\int_0^x g(s)ds - F(0).
\end{aligned}
$$

It follows then that

$$
\begin{aligned}
U(x,t) &= G(x-ct) + F(x+ct) \\
&= \frac{1}{2}f(x-ct) + \frac{1}{2}f(0) - \frac{1}{2c}\int_0^{x-ct} g(s)ds - F(0) \\
&\quad + \frac{1}{2}f(x+ct) - \frac{1}{2}f(0) + \frac{1}{2c}\int_0^{x+ct} g(s)ds + F(0).
\end{aligned}
$$

Simplifying and rearranging one obtains the expression

$$
U(x,t) = \frac{1}{2}\big(f(x+ct) + f(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds, \tag{3.4}
$$

which is known as *D'Alembert's solution*.

**Note.** *Observe that by prescribing initial conditions one obtains a unique solution.*

**Example 3.1.** The wave equation $U_{tt} = c^2 U_{xx}$ with initial position $U(x,0) = \sin x$ and initial velocity $U_t(x,0) = 0$ is

$$
\begin{aligned}
U(x,t) =& \frac{1}{2}[\sin(x+ct) + \sin(x-ct)] \\
=& \sin x \cos(ct).
\end{aligned}
$$

3.2. **Where does the change of variables come from?** To explain the change of variables

$$
u = x - ct, \qquad v = x + ct. \tag{3.5}
$$

one observes that the wave equation can be rewritten as

$$
U_{tt} - c^2 U_{xx} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)U = 0.
$$

Letting

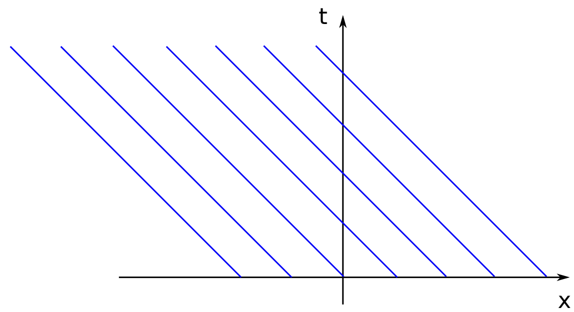
$$
W \equiv \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)U,
$$

then

$$
\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)W = \frac{\partial W}{\partial t} - c\frac{\partial W}{\partial x} = W_t - cW_x = 0.
$$

Thus, $W$ satisfies a first order pde with constant coefficients —we have already studied the solutions to this equation. The characteristics are lines with negative slope $dt/dx = -1/c$ (negative slope) so that

$$
x + ct = \text{constant}.
$$

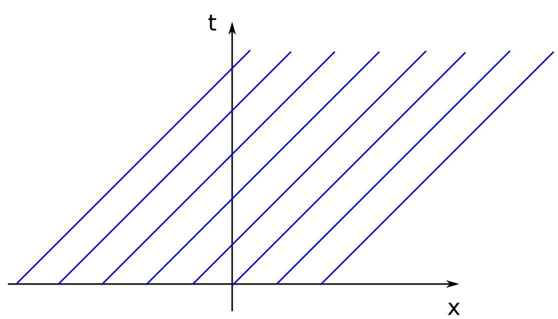



Once we know $W$ one has to solve the equation

$$U_t + cU_x = W$$

which is, again, a first order pde with constant coefficients —observe, however, that the equation is inhomogeneous. The slope of the characteristics is $dt/dx = 1/c$ (positive slope) so that

$$x - ct = \text{constant}.$$

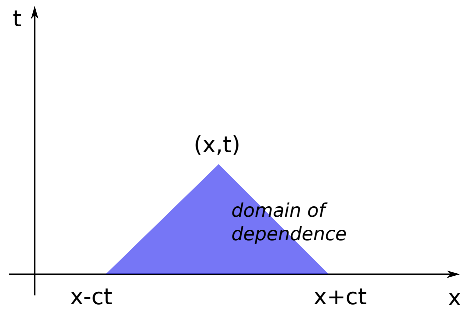



**Note.** *Thus, the wave equation has two sets of characteristics —that is, there is information travelling in two directions: to the left and to the right.*

3.3. **Interpretation of D'Alembert's solution.** Formula (3.4) can be read as saying

$$\begin{aligned} U(x,t) = &\big(\text{average of } U(x,0) \text{ on } x-ct \text{ and } x+ct\big) \\ &+\big(\text{average of } U_t(x,0) \text{ over the interval } [x-ct, x+ct]\big). \end{aligned}$$

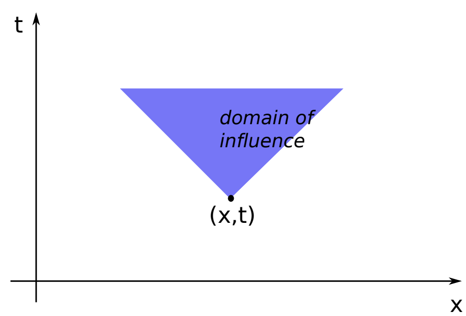
Hence, $U(x,t)$ only depends on the initial conditions on the interval $[x-ct, x+ct]$ —see the figure below.



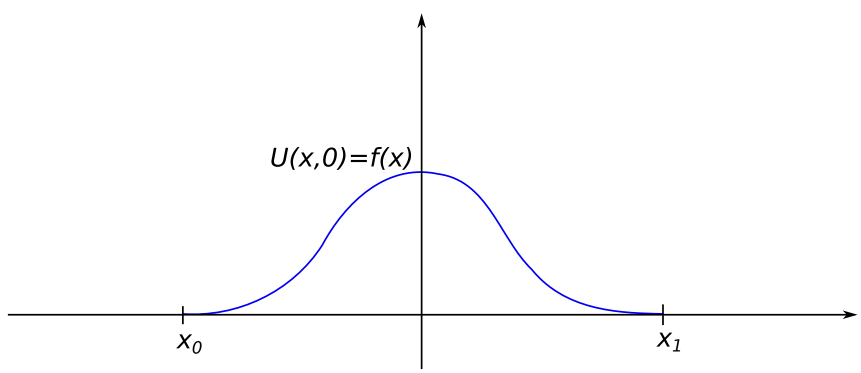

The region in the $(x,t)$ diagram that have an influence in the value of $U(x,t)$ at $(x,t)$ is called the **domain of dependence** of $(x,t)$.

**Note.** *This has connections with Relativity (MTH6132) —information cannot travel at infinite speed.*

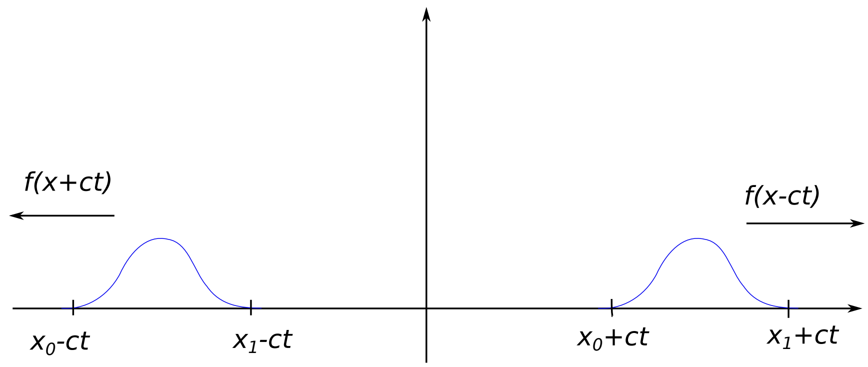
Conversely, given a point $(x,t)$ (event) it influences the region shown below:




This region is called the **domain of influence**.
Also, assume that $g(x) = 0$ and that $f(x)$ has the shape of a bump:




Then, at later times the solution looks like:




That is, one has two bumps, half the size of the initial one moving in opposite directions. The above situation can be described in a diagram in the $(x, t)$ plane as follows: