PS 2: 5

[20 points]

By the change of coodinate

$$\begin{cases} \tilde{x} = x+2y \\ \tilde{y} = 2x-y \end{cases}$$

we have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial x} + \frac{\partial}{\partial \tilde{y}}\frac{\partial \tilde{y}}{\partial x}$$

$$= \frac{\partial}{\partial \tilde{x}} + 2\frac{\partial}{\partial \tilde{y}}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial y} + \frac{\partial}{\partial \tilde{y}}\frac{\partial \tilde{y}}{\partial y}$$

$$= 2\frac{\partial}{\partial \tilde{x}} - \frac{\partial}{\partial \tilde{y}}$$

Now the equation becomes

$$\left(\frac{\partial}{\partial \tilde{x}} + 2\frac{\partial}{\partial \tilde{y}}\right)u + 2\left(2\frac{\partial}{\partial \tilde{x}} - \frac{\partial}{\partial \tilde{y}}\right)u + (2x-y)u = (x+2y)(2x-y)$$

$$(1+4)u_{\tilde{x}} + (2-2)u_{\tilde{y}} + \tilde{y}u = \tilde{x}\cdot\tilde{y}$$

$$5u_{\tilde{x}} + \tilde{y}u = \tilde{x}\cdot\tilde{y}$$

Using integrating factor $e^{\frac{\tilde{x}\tilde{y}}{5}}$

we get $e^{\frac{\tilde{x}\tilde{y}}{5}}u_{\tilde{x}} + \frac{\tilde{y}}{5}e^{\frac{\tilde{x}\tilde{y}}{5}}u = \dfrac{e^{\frac{\tilde{x}\tilde{y}}{5}}\cdot\tilde{x}\tilde{y}}{5}$

$$\frac{\partial}{\partial \tilde{x}}\left[e^{\frac{\tilde{x}\tilde{y}}{5}} u\right] = \frac{1}{5} e^{\frac{\tilde{x}\tilde{y}}{5}} \cdot \tilde{x} \cdot \tilde{y}$$

Integrating with respect to $\tilde{x}$ gives

$$e^{\frac{\tilde{x}\tilde{y}}{5}} u = \int \frac{1}{5} e^{\frac{\tilde{x}\tilde{y}}{5}} \tilde{x}\tilde{y}\, d\tilde{x}$$

Using integration by parts, we get

$$e^{\frac{\tilde{x}\tilde{y}}{5}} u = e^{\frac{\tilde{x}\tilde{y}}{5}} \cdot \tilde{x} - \int e^{\frac{\tilde{x}\tilde{y}}{5}} d\tilde{x}$$

$$e^{\tilde{x}\tilde{y}} u = e^{\frac{\tilde{x}\tilde{y}}{5}} \cdot \tilde{x} - \frac{5}{\tilde{y}} e^{\frac{\tilde{x}\tilde{y}}{5}} + f(\tilde{y})$$

$$u(\tilde{x}, \tilde{y}) = \tilde{x} - \frac{5}{\tilde{y}} + f(\tilde{y}) e^{-\frac{\tilde{x}\tilde{y}}{5}}$$

Changing the coodinate back, we get

$$u(x, y) = x + 2y - \frac{5}{2x-y} + f(2x-y) \cdot e^{\frac{-2x^2-3xy+2y^2}{5}}$$

PS2 : 7

[10 points]

The characteristic curves are

$$\frac{dt}{dx} = -1$$

$$t + x = c$$

So the equation becomes

$$\frac{\partial}{\partial x} u - \frac{\partial t}{\partial x} u - u = 0$$

$$\frac{d}{dx} u = u$$

$$\frac{du}{u} = 1$$

integrating both sides : $\ln u = x + f(c)$

$$u = e^{x} \cdot \hat{f}(c)$$

$$u(x,t) = e^{x} \cdot \hat{f}(t+x)$$

Now when $t = 0$, we get

from the characteristic equation that

$$x = c$$

So $2 = u(x,0) = e^{c} \cdot \hat{f}(c)$

we get $\hat{f}(c) = 2 \cdot e^{-c}$

The solution to the initial value problem is thus

$$u(x,t) = e^{x} \cdot 2 \cdot e^{-c}$$

$$= e^{x} \cdot 2 \cdot e^{-(x+t)}$$

$$= 2e^{-t}$$

Selected Solutions to Problem Set 2.

1. (1). We use the method of change of variables

$$\begin{cases} \tilde{x} = x - 2t \\ \tilde{t} = -2x - t \end{cases}$$

then $u_x = \frac{\partial u}{\partial \tilde{x}} \frac{\partial \tilde{x}}{\partial x} + \frac{\partial u}{\partial \tilde{t}} \frac{\partial \tilde{t}}{\partial x} = u_{\tilde{x}} - 2u_{\tilde{t}}$

$u_t = \frac{\partial u}{\partial \tilde{x}} \frac{\partial \tilde{x}}{\partial t} + \frac{\partial u}{\partial \tilde{t}} \cdot \frac{\partial \tilde{t}}{\partial t} = -2u_{\tilde{x}} - u_{\tilde{t}}$

The equation then becomes

$$(u_{\tilde{x}} - 2u_{\tilde{t}}) - 2(-2u_{\tilde{x}} - u_{\tilde{t}}) = 0$$

Namely $5u_{\tilde{x}} = 0$

Integrate ~~$u_{\tilde{x}} = f$~~ and get

$$\boxed{u = \cancel{f(\tilde{t})}\, f(\tilde{t}) = f(-2x-t)}$$

2. (1). It's the same equation as in 1(1), we ~~use~~ use the method of characteristic here.

the characteristic lines are,

$$\cancel{\frac{dt}{dx}}\ \frac{dt}{dx} = \frac{-2}{1}.$$

so they are straight lines

$$\boxed{t = -2x + C} \quad \text{or} \quad \boxed{2x + t = C}$$

The solution only depends on ~~the~~ which characteristic line it's on, so

$$u(x,t) = \hat{f}(C) = \hat{f}(2x+t) = f(-2x-t)$$

where $\boxed{\hat{f}(s) = f(-s).}$

Now we specify f by the ~~initial~~ boundary condition.
when $x=0$, $t=C$, and so

$$\cos C = \cos t = u(0,t) = f(-t) = f(-c)$$

$$\text{so } f(c) = \cos(-c) \text{ and } f(-c) = \cos c$$

and the ~~general~~ solution is

$$\underline{u(x,t)} = f(-2x-t) = f(-c) = \cos C$$

$$= \underline{\cos(2x+t)}$$

4. The change of variable we use is

$$\begin{cases} \tilde{x} = ax+by \\ \tilde{y} = bx-ay. \end{cases}$$

so

$$\begin{cases} u_x = a u_{\tilde{x}} + b u_{\tilde{y}} \\ u_y = b u_{\tilde{x}} - a u_{\tilde{y}} \end{cases}$$

The Left hand side of equation becomes

$$\cancel{a u_x +}\; a(au_{\tilde{x}} + bu_{\tilde{y}}) + b(bu_{\tilde{x}} - au_{\tilde{y}}) + cu = 0$$

Namely

$$(a^2+b^2) u_{\tilde{x}} + cu = 0$$

$$u_{\tilde{x}} + \frac{c}{a^2+b^2} u = 0.$$

Using integrating factor $\left(e^{\frac{c}{a^2+b^2}\tilde{x}}\right)$, we have

$$\frac{\partial}{\partial \tilde{x}}\left(e^{\frac{c}{a^2+b^2}\tilde{x}} u\right) = 0$$

$$e^{\frac{c}{a^2+b^2}\tilde{x}} u = f(\tilde{y})$$

③.

$$u = f(\tilde{y}) \cdot e^{-\frac{c}{a^2+b^2}\tilde{x}}$$

$$= f(bx-ay) \cdot e^{-\frac{c}{a^2+b^2}(ax+by)}$$

is the general solution

6. Do change of variable

$$\begin{cases} \tilde{x} = x+y \\ \tilde{y} = x-y \end{cases}$$

we get $\begin{cases} x = \frac{1}{2}(\tilde{x}+\tilde{y}) \\ y = \frac{1}{2}(\tilde{x}-\tilde{y}) \end{cases}$

LHS of the equation becomes

$$u_x + u_y + u = \left(u_{\tilde{x}} \cdot \frac{\partial \tilde{x}}{\partial x} + u_{\tilde{y}} \frac{\partial \tilde{y}}{\partial x}\right) + \left(u_{\tilde{x}} \frac{\partial \tilde{x}}{\partial y} + u_{\tilde{y}} \frac{\partial \tilde{y}}{\partial y}\right) + u$$

$$= \cancel{2u_{\tilde{x}} + 2u} \quad u_{\tilde{x}} + u_{\tilde{y}} + u_{\tilde{x}} - u_{\tilde{y}} + u$$

$$= 2u_{\tilde{x}} + u$$

RHS of the equation is

$$e^{x+2y} = e^{\frac{1}{2}(\tilde{x}+\tilde{y}) + (\tilde{x}-\tilde{y})} = e^{\frac{3\tilde{x}}{2} - \frac{\tilde{y}}{2}}$$

The equation becomes

$$2u_{\tilde{x}} + u = e^{\frac{3\tilde{x}}{2} - \frac{\tilde{y}}{2}}$$

Integrate ~~we get~~ Using the integrating factor $e^{\frac{\tilde{x}}{2}}$, we get

$$\frac{\partial}{\partial \tilde{x}}\left(e^{\frac{\tilde{x}}{2}} u\right) = \frac{1}{2} e^{-\frac{1}{2}\tilde{y}} \cdot e^{2\tilde{x}}$$

$$e^{\frac{\tilde{x}}{2}} \cdot u = \frac{1}{2} \int e^{2\tilde{x}} \cdot e^{-\frac{1}{2}\tilde{y}} \, d\tilde{x}$$

$$u = \frac{1}{2} e^{-\frac{\tilde{x}}{2}} \cdot e^{-\frac{\tilde{y}}{2}} \cdot \left[\frac{1}{2} e^{2\tilde{x}} + f(\tilde{y})\right]$$

$$u(x, y) = \frac{1}{4} e^{\frac{3\tilde{x}}{2} - \frac{\tilde{y}}{2}} + \frac{1}{2} e^{-\frac{\tilde{x}}{2} - \frac{\tilde{y}}{2}} \cdot f(\tilde{y})$$

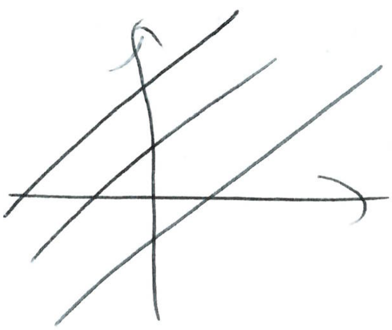
$$= \frac{1}{4} e^{x+2y} + \frac{1}{2} e^{-\frac{x+y}{2}} f(x-y)$$

8. The characteristic satisfy the ODE

$$\frac{dy}{dx} = 1.$$

the charateristic curves are then

$$y = x + C$$



The equation becomes

$$u_x + u_y$$

$$= u_x + \frac{y}{x} u_y$$

$$= u_x + \frac{\partial y}{\partial x} u_y$$

$$= \frac{d}{dx} u$$

$$= 1$$

so $u(x,y) = x + f(c)$

$= x + f(y-x).$

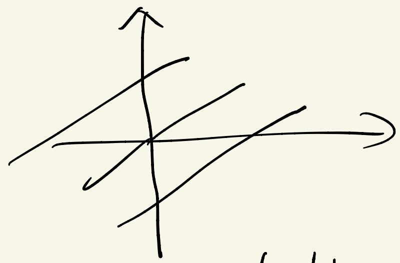
Selected solutions to Problem Set 2 (continued)

2 (2) This is an initial value problem (IVP)

step 1. The characteristic lines are given by

$bx - ay = c$ with $a = \pi$, $b = 1$

i.e. $x - \pi y = c$ (characteristic line)



so the general solution is

$u(x, y) = f(c) = f(x - \pi y)$, for any function f.
(general solution)

step 2. Next we specify f using the initial value

$f(c) = u(x, 0) = x^2$ when $y = 0$

on the on the other hand,
when $y = 0$, we get from the characteristic line equation

that $x = c$

so $f(c) = u(x,0) = x^2 = c^2$

plug into the expression of f to the general solution, we get

$u(x,y) = (x-\pi y)^2$

3. (1) $\frac{\partial}{\partial x}(x+e^t) = 1$

$\frac{\partial}{\partial t}(x+e^t) = e^t$

and $\frac{\partial}{\partial x}(x+e^t) + \frac{\partial}{\partial t}(x+e^t) = 1+e^t$

so it is a solution.

(2). Step 1: solve the general solution for the homogeneous equation

$\tilde{u}_x + \tilde{u}_t = 0$,

get $\tilde{u}(x,t) = f(x-t)$, for any f

step 2: By the principle of superposition, and combining step 1, the general solution to the inhomogeneous equation is

$u(x,t) = 1+e^t + f(x-t)$, for any f

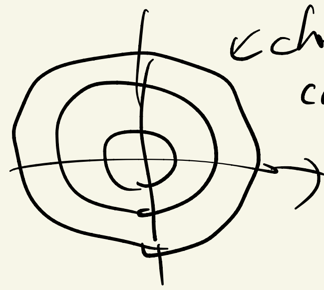
PS2 Q9:

The characteristics are $\frac{dt}{dx} = \frac{x}{-t}$

i.e. $-t\,dt = x\,dx$

$$-\frac{t^2}{2} = \frac{x^2}{2} + \tilde{c}$$

$$t^2 + x^2 = c$$



Along the characteristics, the PDE becomes an ODE

$$\frac{d}{dx} u(x, t(x)) = u_x - \frac{dt}{dx} u_t$$

$$= u_x - \frac{x}{-t} u_t$$

$$= \frac{t u_x - x u_t}{t}$$

$$= \frac{-1}{t}\left[x u_t - t u_x\right]$$

$$= -\frac{1}{t}$$

↑ using the PDE

using $x^2 + t^2 = c$

we get $t = \sqrt{c - x^2}$

So the ODE is

$$\frac{du}{dx} = \frac{-1}{\sqrt{c - x^2}}$$

Solve it we get

$$u = -\arcsin\frac{x}{\sqrt{C}} + f(C)$$

using that $C = t^2+x^2$ for the characteristic equations. we get

$$u(x,t) = -\arcsin\frac{x}{\sqrt{t^2+x^2}} + f(x^2+t^2)$$

$$= -\arctan\left(\frac{x}{t}\right) + f(x^2+t^2)$$

using the boundary conditions, we have

$$0 = u(0,t) = -\arctan 0 + f(t^2)$$

we get $f(t^2) = 0$

replacing t by $\sqrt{t}$, we get

$$f(t) \equiv 0$$

So the solution to the BVP is

$$u(x,t) = -\arctan\left(\frac{x}{t}\right).$$