MTH6106 Group Theory – Coursework 4

December 6, 2023

This coursework counts for 4% of your mark for this module. You should answer all questions, and each question will be marked out of 4. You should give full explanation of your answers. Please submit your solutions on QMPlus by 2pm on Friday 2nd December. Your submission must be entirely your own work.

1. Prove, by obtaining a contradiction, that there cannot exist a homomorphism $\psi\colon Q_8 \to S_5$ such that $\psi(j) = (1234)$ and $\psi(k) = (5314)$.

   If such a homomorphism exists then we would have $\psi(jk) = (1234)(5314) = (23)(45)$ and $\psi(kj) = (5314)(1234) = (12)(35)$. There are several ways of deriving a contradiction from this. Firstly, in Q_8 we have $jkkj = i(-i) = -i^2 = 1$ so $\psi(jkkj)$ must be the identity, which is not the case here since $(23)(45)(12)(35) = (13452) \neq \text{id}$. Secondly, if $\psi(jk) = (1234)(5314) = (23)(45)$ then in particular $\psi(i) = \psi(jk)$ has order 2, so $\psi(i^2)$ must be the identity, in which case we should have $\psi(jk) = \psi(-1)\psi(jk) = \psi(-jk) = \psi(kj)$ which is false. This list of routes to a contradiction is not exhaustive.

2. Write down a homomorphism $\varphi\colon \mathcal{C}_2 \times \mathcal{C}_2 \to \mathcal{C}_2 \times \mathcal{C}_2$ such that $\operatorname{im} \phi = \ker \phi$. Can there exist a homomorphism $\varphi\colon D_8 \to D_8$ with the same property? Why, or why not?

   There are several ways of doing this. One is to define $\varphi(g, h) := \varphi(gh, gh)$ for all $(g, h) \in \mathcal{C}_2 \times \mathcal{C}_2$. To see that it is a homomorphism note that

   $$\varphi(g_1g_2, h_1h_2) = (g_1g_2h_1h_2, g_1g_2h_1h_2) = (g_1h_1g_2h_2, g_1h_1g_2h_2) = \varphi(g_1, h_1)\varphi(g_2, h_2)$$

   for every $(g_1, h_1), (g_2, h_2) \in \mathcal{C}_2 \times \mathcal{C}_2$. Both the kernel and the image are $\{(g, g)\colon g \in \mathcal{C}_2\}$. Some alternatives which also work are $\varphi(g, h) := (h, 1)$ and $\varphi(g, h) := (1, g)$.

   Any homomorphism $\varphi\colon G \to G$ must satisfy $|G|/|\ker \varphi| = |\operatorname{im} \varphi|$ by the First Isomorphism Theorem, so if $|\ker \varphi| = |\operatorname{im} \varphi|$ then necessarily $|\ker \varphi| = \sqrt{|G|}$. If $|G| = 8$ this is impossible since $|\ker \varphi|$ would have to be non-integer.

3. Find all the automorphisms of C_{10} and write down a Cayley table for $\mathrm{Aut}(C_{10})$.

   Write $C_{10} = \{1, z, z^2, \ldots, z^9\} \le \mathbb{C}^\times$ where $z^{10} = 1$. If $\phi\colon C_{10} \to C_{10}$ is a homomorphism then it is completely determined by $\phi(z)$, since then $\phi(z^k) = \phi(z)^k$ for all $k = 0, \ldots, 9$. On the other hand, if $0 \le m < 10$ then defining $\phi_m(z^k) := z^{mk}$ for every $k = 0, \ldots, 9$ defines a homomorphism $\phi_m\colon C_{10} \to C_{10}$. It follows that there are exactly 10 *homomorphisms* from C_{10} to itself.

   Now, a homomorphism from a group to itself is an automorphism if and only if it is bijective. Since ϕ_m is a function from a finite set to itself, it is bijective if and only if it is injective. The homomorphism ϕ_m is injective if and only if the only solution to $\phi_m(z^k) = 1$ is $k = 0$, if and only if the equation $km = 0 \mod 10$ has no solutions except $k = 0 \mod 10$, if and only if m has no common factors with 10. So ϕ_m is an automorphism if and only if m has no common factors with 10. There are therefore four elements in $\mathrm{Aut}(C_{10})$: ϕ_1, ϕ_3, ϕ_7, ϕ_9.

   The Cayley table is

	ϕ_1	ϕ_3	ϕ_7	ϕ_9
ϕ_1	ϕ_1	ϕ_3	ϕ_7	ϕ_9
ϕ_3	ϕ_3	ϕ_9	ϕ_1	ϕ_7
ϕ_7	ϕ_7	ϕ_1	ϕ_9	ϕ_3
ϕ_9	ϕ_9	ϕ_7	ϕ_3	ϕ_1

   .

4. Lemma 5.6 tells us that the order of every conjugacy class divides the order of the group, which in this case is 14. So every conjugacy class has size 1, 2, 7 or 14. It is easily checked from the definition (or by Lemma 3.8) that 1 is in its own conjugacy class, $\{1\}$, so all other conjugacy classes must have size 1, 2, or 7.

   Now we can check that $rsr^{-1} = rrs = r^2s$, $r(r^2s)r^{-1} = r^4s$, $r(r^4s)r^{-1} = r^6s$, and so on, and it follows easily that the conjugacy class of s includes $\{r^ks\colon 0 \le k < 7\}$. Therefore the order of the conjugacy class of s is *at least* 7, but by the previous reasoning this implies that it is *exactly* seven. So $\{s, rs, r^2s, \ldots, r^6s\}$ is a conjugacy class.

   This leaves only the six nontrivial rotations to account for, and by elimination their conjugacy classes must have size either 1 or 2. It is easy to check that $srs^{-1} = srs = ssr^{-1} = r^{-1} = r^6$ so r and r^6 are conjugate, hence one of the classes is $\{r, r^6\}$. Similarly $sr^2s^{-1} = r^5$ and $sr^3s^{-1} = r^4$. The classes therefore must be $\{1\}$, $\{r, r^6\}$, $\{r^2, r^5\}$, $\{r^3, r^4\}$ and $\{s, rs, r^2s, \ldots, r^6s\}$.

5. Consider a nonzero vector $(u, v)^T \in \mathbb{F}_p^2$. If $u \ne 0$ then the matrix

$$\begin{pmatrix} u & 0 \\ v & u^{-1} \end{pmatrix}$$

belongs to $\mathrm{SL}_2(\mathbb{F}_p)$ and satisfies

$$\begin{pmatrix} u & 0 \\ v & u^{-1} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}.$$

If $v \neq 0$ then similarly

$$\begin{pmatrix} u & -v^{-1} \\ v & 0 \end{pmatrix}$$

belongs to $\mathrm{SL}_2(\mathbb{F}_p)$ and satisfies

$$\begin{pmatrix} u & -v^{-1} \\ v & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}.$$

In either case $(u, v)^T$ belongs to the orbit as required. The orbit therefore contains $p^2 - 1$ vectors. On the other hand the stabiliser consists of all matrices of the form

$$\begin{pmatrix} 1 & c \\ 0 & d \end{pmatrix}$$

with determinant 1, which is to say, all matrices of the form

$$\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$$

and there are p such matrices. By the orbit-stabiliser theorem it follows that $|\mathrm{SL}_2(\mathbb{F}_p)| = p(p^2 - 1)$.

6. All eight faces of this solid are the same identical isoceles triangle, so by rotation we can see that the orbit of any one face is just the set of all possible faces. The orbit of a face therefore has eight elements. The *stabiliser* of a face contains the identity, and also contains a single reflection in the plane which bisects the face. Therefore $|G| = 16$.

   You can verify this calculation by instead considering the action on vertices. This is made more complicated by the fact that there are two "types" of vertex: the two "top and bottom" vertices which each meet four "long" edges, and the "middle" vertices which each meet two "long" edges and two "short" edges. The orbit of the first type of vertex has two elements, and the stabiliser consists of all symmetries of the square base, so we get $2 \times 8 = 16$. The orbits and stabilisers of "middle" vertices have four and four elements respectively.