# MTH6106 Group Theory – Coursework 4

December 6, 2023

This coursework counts for 4% of your mark for this module. You should answer all six questions, and each question will be marked out of 4. You should give full explanation of your answers. Please submit your solutions on QMPlus by 2pm on Friday 1st December. Your submission must be entirely your own work.

1. Prove directly that there cannot exist a homomorphism $\psi\colon Q_8 \to S_5$ such that $\psi(j) = (1234)$ and $\psi(k) = (5314)$.

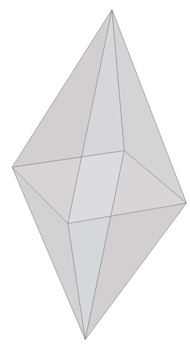
2. Write down a homomorphism $\varphi\colon \mathcal{C}_2 \times \mathcal{C}_2 \to \mathcal{C}_2 \times \mathcal{C}_2$ such that $\operatorname{im} \phi = \ker \phi$. Can there exist a homomorphism $\varphi\colon D_8 \to D_8$ with the same property? Why, or why not?

3. Find all the automorphisms of $C_{10}$ and write down a Cayley table for $\mathrm{Aut}(C_{10})$.

4. Find all of the conjugacy classes of elements of $D_{14}$, making use of Lemma 5.6 to simplify your argument. (You may also find Lemma 3.8 helpful.)

5. Let $p$ be a prime number and let $\mathbb{F}_p$ denote the field with $p$ elements, $\mathbb{F}_p = \{0, 1, \ldots, p-1\}$, equipped with the operations of arithmetic modulo $p$. Consider the action of $\mathrm{SL}_2(\mathbb{F}_p)$ on $\mathbb{F}_p^2$ defined by $\pi_A(v) := Av$. Show that the *orbit* of the vector $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ is the set of all nonzero vectors in $\mathbb{F}_p^2$. How many elements belong to the *orbit* and *stabiliser* of this vector? Use these answers to calculate $|\mathrm{SL}_2(\mathbb{F}_p)|$.

6. Consider a solid shape $S$ which consists of two identical tall, square-based pyramids joined at the base as illustrated below:



Each side of this pyramid is an isoceles triangle which is not equilateral.

Let $G$ denote the group of symmetries of $S$. Calculate $|G|$ using the orbit-stabiliser theorem. (You are advised to proceed by considering the orbit and stabiliser of a *face* of $S$, but you could consider the action on edges or vertices if you prefer.)