MTH6106 Group Theory – Coursework 3

November 7, 2023

This coursework counts for 4% of your mark for this module. You should answer all questions, and each question will be marked out of 4. You should give full explanation of your answers. Please submit your solutions on QMPlus by 2pm on Friday 10th November. Your submission must be entirely your own work.

1. Write down the Cayley table for the quotient group $\mathcal{D}_{12}/\langle r^2 \rangle$. (You do not need to prove that $\langle r^2 \rangle$ is normal.)

2. Let $d \geq 1$, let $\mathbb{F}$ be a field and let $\mathrm{Aff}_d(\mathbb{F})$ denote the set of all functions $T_{A,v} \colon \mathbb{F}^d \to \mathbb{F}^d$ which have the form $T_{A,v}(x) = Ax + v$, where A is an invertible $d \times d$ matrix with entries in $\mathbb{F}$ and where $v \in \mathbb{F}^d$. Prove that $\mathrm{Aff}_d(\mathbb{F})$ is a group (where the binary operation is given by composition of functions).

3. Let $d \geq 1$ and let $\mathbb{F}$ be a field, and define a group of matrices $G \leq \mathrm{GL}_{d+1}(\mathbb{F})$ by
$$G = \left\{ \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} : A \in \mathrm{GL}_d(\mathbb{F}), v \in \mathbb{F}^d \right\}.$$
Prove that G is isomorphic to the group
$$\mathrm{Aff}_d(\mathbb{F}) = \{T_{A,v} \colon A \in \mathrm{GL}_d(\mathbb{F}), v \in \mathbb{F}^d\}$$
considered in the previous question. (You do not need to prove that G is a group.)

4. For every integer $r \geq 1$ let $\mathcal{C}_r$ denote the cyclic group
$$\{e^{2\pi i k/r} \colon 0 \leq k < r\} \leq \mathbb{C}^\times.$$
Let $n, n \geq 1$ be a pair of integers whose highest common factor is 1. Define a function $\varphi \colon \mathcal{C}_n \times \mathcal{C}_m \to \mathcal{C}_{nm}$ by
$$\varphi(e^{2\pi i k/n}, e^{2\pi i \ell/m}) := e^{2\pi i k/n} \cdot e^{2\pi i \ell/m}.$$
Prove that φ is an isomorphism.

5. Let $\mathbb{F}$ be a field, and consider the group

$$G := \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{F} \right\} \leq \mathrm{GL}_3(\mathbb{F}).$$

Find the centre $Z(G)$ of the group G. (You do not need to prove that G is a group.)

6. Let G and H be groups, and let N be a normal subgroup of G. Decide whether each of the following two statements is true or false. Give a proof or a counterexample in each case.

(a) The group $G \times H$ is abelian if and only if G and H are both abelian.

(b) The group G/N is abelian if and only if both G and N are abelian.