MTH6016 Group Theory – Coursework 2

November 1, 2023

This coursework counts for 4% of your mark for this module. You should answer all questions, and each question will be marked out of 4. You should give full explanation of your answers. Please submit your solutions on QMPlus by 2pm on Friday 27th October. Your submission must be entirely your own work.

1. Let a be the last nonzero digit of your student number. Find the subgroup $\langle sr^a, sr^{a+2}\rangle \leq \mathcal{D}_8$.

2. Recall that $\mathbb{F}_3$ is the set $\{0, 1, 2\}$ equipped with multiplication modulo 3, with respect to which $\mathbb{F}_3$ is a field. Let G be the group of 2×2 upper-triangular invertible matrices with entries in $\mathbb{F}_3$. The group G has exactly 12 elements. Find all *left* cosets and all *right* cosets of the subgroup

$$H := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right\} \leq G.$$

   You are not required to show that G is a group, and you are also not required to show that H is a subgroup of G.

3. Consider two matrices given by

$$g = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

   Decide whether or not $g \sim_G h$ in each of the following four cases:

   (i) $G = \mathrm{GL}_2(\mathbb{R})$

   (ii) $G = \mathrm{SL}_2(\mathbb{R})$

   (iii) $G = \mathrm{SL}_2(\mathbb{C})$

   (iv) $G = \mathrm{GL}_2(\mathbb{F}_3)$.

4. Let G be a group of order 14. Using Lagrange's theorem, show that every subgroup of G either contains an element of order 2, or contains an element of order 7, or is the trivial subgroup $\{1\}$. Now show that if a subgroup contains an element of order 2 *and* an element of order 7 then that subgroup is equal to G.

5. Let G be a group, let $f, g \in G$, and suppose that $f \sim_G g$. Show that the order of f is equal to the order of g.

6. Let X be a set with exactly n elements, where $n \geq 2$, and consider the group $\mathcal{P}(X)$ with the symmetric difference operation $A \circ B := A \triangle B$ as in Coursework 1.

   (i) Show that there exist n elements $g_1, \ldots, g_n$ such that $\langle g_1, g_2, \ldots, g_n \rangle$ is equal to $\mathcal{P}(X)$. (You do not have to give a detailed proof that your reasoning is correct, but in your answer you should describe a list of n elements sufficient to generate the group.)

   (ii) Prove that there do *not* exist $n-1$ elements $h_1, \ldots, h_{n-1}$ such that $\langle h_1, \ldots, h_{n-1} \rangle = \mathcal{P}(X)$.