Chaos & Fractal

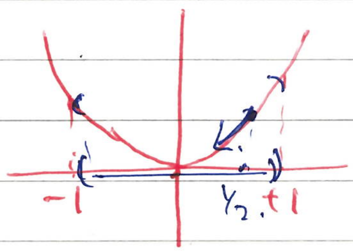
Definition: Let p be a fixed point of $f : \mathbb{R} \to \mathbb{R}$.

$f(p) = p$.

The basin of attraction of p is defined to be the set

$$\underset{\Psi}{\text{Basin}}(p) := \left\{ x_0 \in \mathbb{R} \;\middle|\; \lim_{n \to \infty} f^{(n)}(x_0) = p \right\}$$

Example: $f(x) = x^2$.



$$\left[\begin{array}{l} x^2 = x. \quad \text{fixed pts.} \\ \Rightarrow x(x-1) = 0 \\ \Rightarrow x = 0, 1. \end{array} \right]$$

$$\text{Basin}(0) := \left\{ x_0 \in \mathbb{R} \;\middle|\; \lim_{n \to \infty} f^n(x_0) = 0 \right\}.$$

$$f(x_0) = x_0^2, \quad f^2(x_0) = x_0^4, \quad f^3(x_0) = f(f^2(x_0)).$$
$$= x_0^8$$

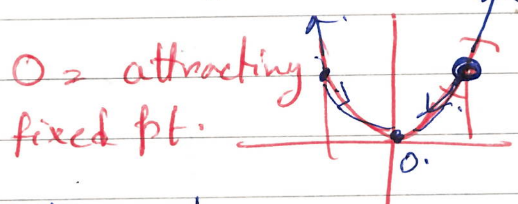
$$f^n(x_0) = x_0^{2^n}$$

$$\left\{ x_0 \in \mathbb{R} \;\middle|\; \lim_{n \to \infty} x_0^{2^n} = 0 \right\}. = \{ x_0 \mid -1 < x_0 < 1$$

$$\left[\begin{array}{llll} \lim\limits_{m \to \infty} x^m & = 0 & \text{if} & |x| < 1 \\ & = 1 & \text{if} & x = 1 \\ & = \infty & & x > 1 \end{array} \right]$$

$$\text{Basin}(1) := \left\{ x_0 \in \mathbb{R} \;\middle|\; \lim_{n\to\infty} \underbrace{f^n(x_0)}_{\substack{\shortparallel \\ x_0^{2^n}}} = 1 \right\}.$$

$$= \{1, -1\}$$

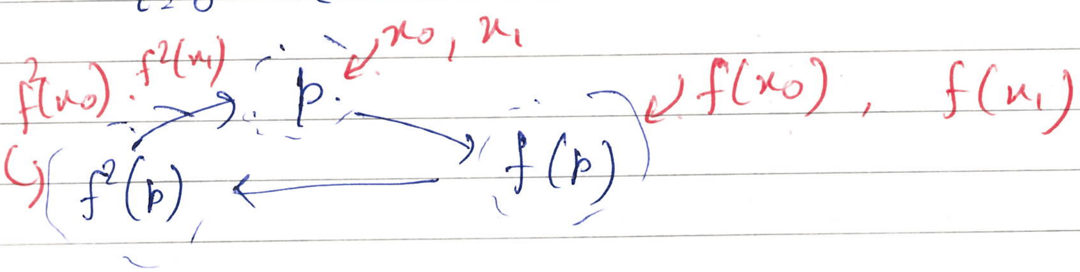


We can extend the definition of basin of attraction to periodic orbits.

Def: Let p be a periodic point of least period k, under the map f. Then the basin of attraction of the k-cycle: $\{p, f(p), f^2(p), \dots, f^{k-1}(p)\}$. is the set.

$$\text{Basin}\left(\{p, f(p), \dots, f^{k-1}(p)\}\right)$$

$$= \bigcup_{i=0}^{k-1} \left\{ x_0 \in \mathbb{R} \;\middle|\; \lim_{n\to\infty} f^{nk}(x_0) = f^i(p) \right\}.$$



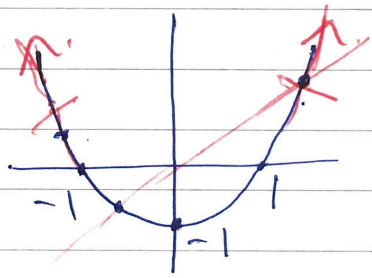
Example: Consider $f : \mathbb{R} \to \mathbb{R}$

$$x \longmapsto x^2 - 1.$$

Fixed points := $x^2 - 1 = x$.

$$x^2 - x - 1 = 0.$$

$$x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2} \approx \begin{matrix} 1.62 \leftarrow x^+ \\ -0.62 \leftarrow x^- \end{matrix}$$



$1.62 \to$ ~~attra~~ "repelling" fixed point.

$-0.62 \to$

Q: Can any numbers from (x^+, ∞) $(-\infty, -x^+)$ be inside Basin (x^+) ?

Note that
$$f^2(x) = (x^2-1)^2 - 1$$
$$= x^4 - 2x^2$$
$$f^3(x) = (x^4 - 2x^2)^2 - 1$$
$$= x^8 + \dots - 1$$

$$\lim_{n \to \infty} f^n(x) = \infty \quad \text{if} \quad x > 1.62 = x$$
$$x < -1.62.$$

$$f^4(x) = (x^8 + \dots - 1)^2 - 1$$
$$= x^{16} + \dots + 0$$

Punchline: It turns out that there is a 2-cycle consisting of $\{0, -1\}$.

$x_0 = -\frac{1}{2}$. $f(x_0) = -0.75$

$f^2(x_0) = -0.4375$

$f^3(x_0) = -0.8085$

$f^4(x_0) = -0.345$

$f^5(x_0) = -0.8801$.

$f^6(x_0) = -0.225$

$f^7(x_0) = -0.949$.

$f^8(x_0) = -0.098$

$f^9(x_0) = -0.9902$.

$f^{10}(x_0) = -0.0194$

$-\frac{1}{2} \in \text{Basin}(\{-1,0\})$.

Exercise $\text{Basin}(\{-1,0\}) = (-x^+, x^+) \setminus \{x^-\}$

Def: Let p be a fixed point of $f: \mathbb{R} \to \mathbb{R}$. We say that p is **attracting** if there exists $\delta > 0$ such that $(p-\delta, p+\delta) \subset \text{Basin}(p)$.

in other words, $\forall x \in (p-\delta, p+\delta)$,

$$\lim_{n\to\infty} f^n(x) = p.$$

Def: The fixed point p is called repelling if there exists $\delta > 0$ such that if $I = (p-\delta, p+\delta)$, then for all $x \in I \setminus \{p\}$ there exist N such that $f^N(x) \notin I$.

$f(x) = x^2$. Let $\delta = 0.001$, $x = 1.00001$

$$(1.00001)^{2^{1000}} \notin (1-0.001, 1+0.001).$$

In other words, p is repelling if there exists an open interval I centered at p for which every point in $I \setminus \{p\}$ escapes from the interval (under the iteration of f).

Example: $f(x) = ax$ where $a \in \mathbb{R}$.

Fixed points: $ax = x. \Rightarrow$ ~~ax~~

$x(a-1) = 0$

$x = 0$ or $a = 1$.

$\boxed{a \neq 1}$ Determine if 0 is attracting / repelling!

$$f^2(x) = f(f(x)) = f(ax) = a^2 x.$$
$$f^3(x) = f(f^2(x)) = f(a^2 x) = a^3 x.$$
$$\cdots \quad f^n(x) = a^n x.$$

Ex 2 $f^n(x) = a^n x.$

We want $(-\delta, \delta)$ such that $\forall x \in (-$

$\lim_{n\to\infty} f^n(x) = 0 \iff \lim_{n\to\infty} a^n x = 0.$

If $|a| < 1$ then $\lim_{n\to\infty} a^n x = 0 \quad \forall x$

$\Rightarrow \mathbb{R} = \text{Basin}(0).$ [0 is attracti

If $|a| > 1$ then $\lim_{n\to\infty} a^n x = \infty, x$
$= -\infty, x$

For any $(-\delta, \delta) \setminus \{0\}$ there will be N

s.t. $a^n x \notin (-\delta, \delta)$. for $n > N$

[0 is repelling]

~~Ex 3 $f(x) = x^3$~~ What about $a = 1$ ca

$f^n(x) = x$.

In this case any $p \in \mathbb{R}$ is a fixed point.

$\text{Basin}(p) = \{ x \in \mathbb{R} \mid \lim_{n\to\infty} \underbrace{f^n(x)}_{x} = p \} = x.$

$= \{p\}.$

Note that there is no $\delta > 0$. s.t.

$(p-\delta, p+\delta) \subseteq \text{Basin}(p). \Rightarrow$ p is NO attracti

On the other hand, for any $\delta > 0$

$\forall x \in (p-\delta, p+\delta) \setminus \{p\}$ we have

$f^n(x) = x \in (p-\delta, p+\delta). \Rightarrow$ p is NOT repellin