CHAOS & FRACTALS ①

WEEK 2 - LESSON 1

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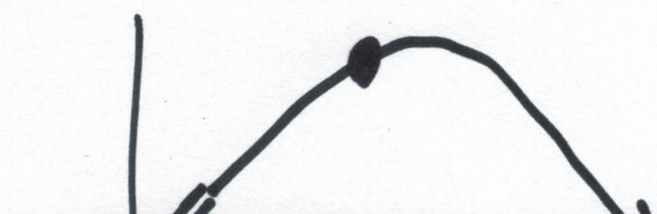
General idea: chaos & fractals related to dynamical systems

↓

things moving.

??

physics.

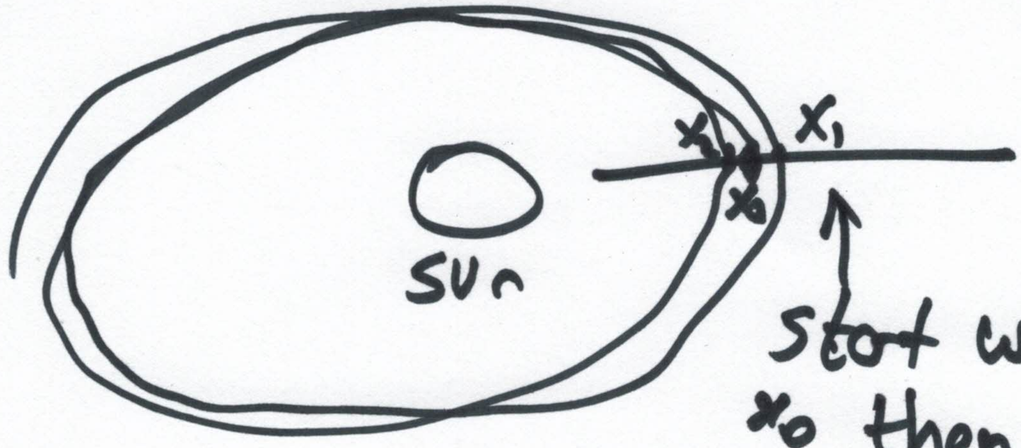


ballistics

To simplify:

- discrete times $x_0, x_1, x_2, \ldots$
- one dimension.

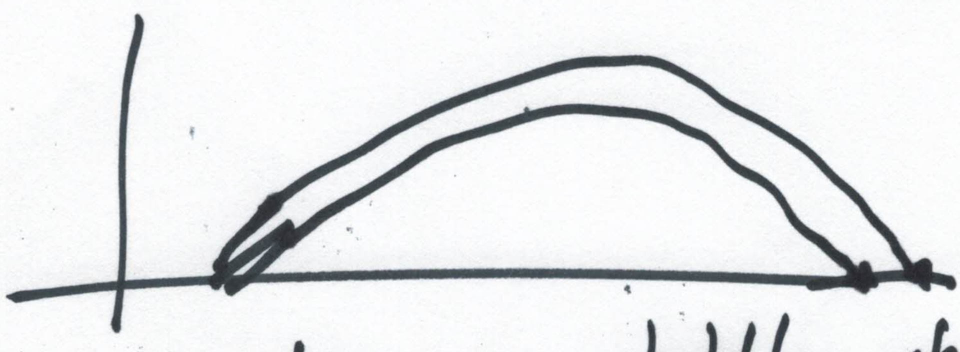
Example



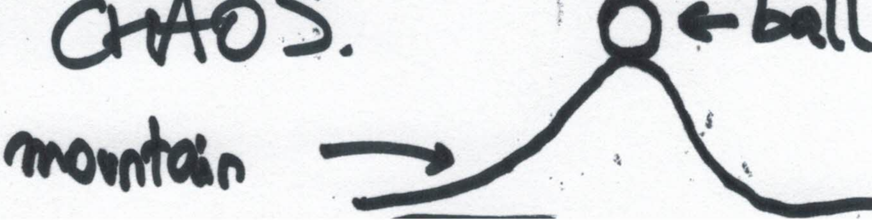
start with x_0 then $f(x_0)$ then $f(f(x_0))$ ...

In this course we start looking at things like

→ If changing initial conditions a little bit, changes trajectory a little bit → STABLE



→ But sometimes little changes in initial conditions change trajectory a lot ⇒ CHAOS.

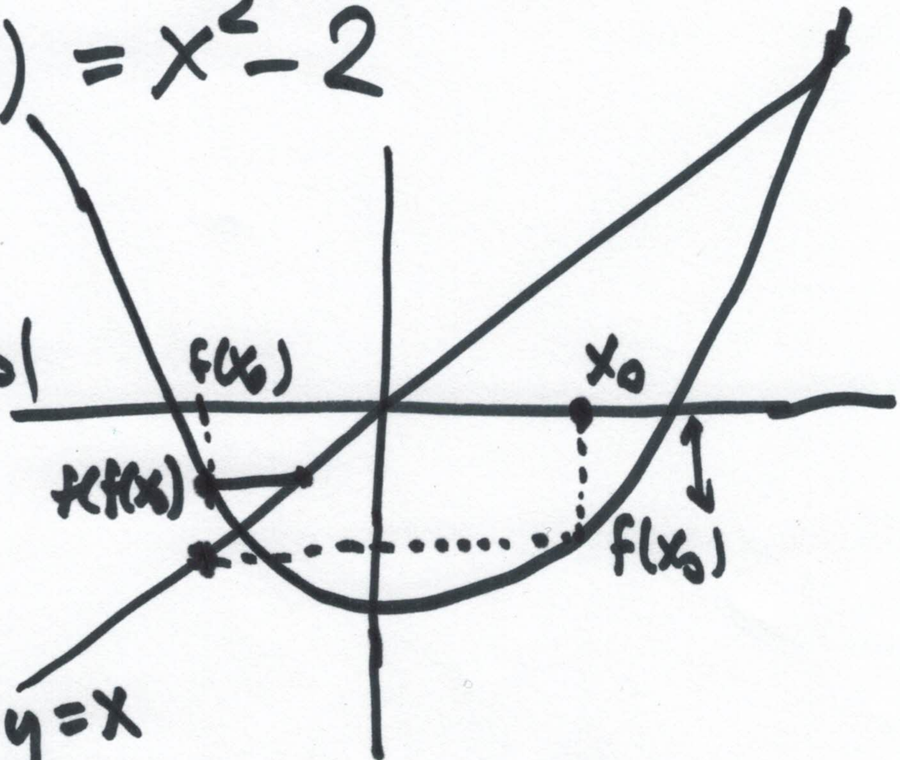


This is why we study

$$x_0 \longrightarrow f(x_0) \longrightarrow \underset{\overset{\shortparallel}{f^2(x_0)}}{f(f(x_0))} \longrightarrow \underset{\overset{\shortparallel}{f^3(x_0)}}{f(f(f(x_0)}$$

<u>Example</u> $f(x) = x^2 - 2$

How does this dynamical system behave?



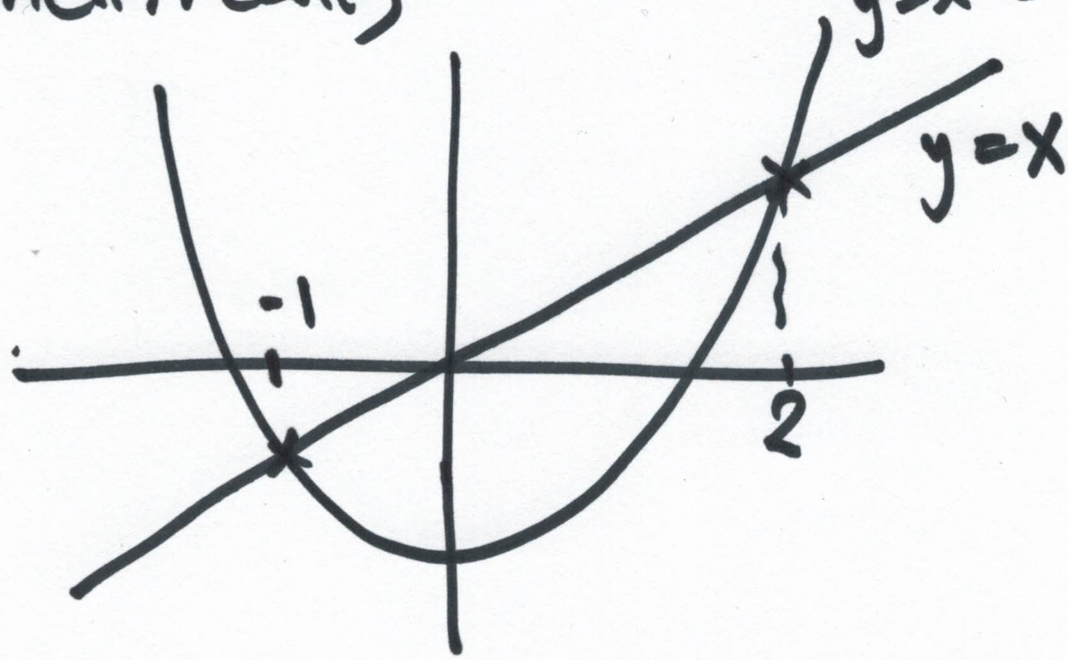
Fixed points : $f(x) = x$

solve

$$x^2 - 2 = x \;,\; x^2 - x - 2 = 0$$

$$x = \frac{1 \pm \sqrt{1 + 4 \cdot 2}}{2} = \begin{cases} 2 \\ -1 \end{cases}$$

Geometrically



So if $x_0=-1$, then $f(x_0)=(-1)^2-2$
$=-1$

<u>Other interesting values</u>
are "eventually fixed" points

① $x_0=0$, $f(x_0)=(0)^2-2=-2$

$f^2(x_0)=f(-2)=(-2)^2-2$
$=2$

pictorially

$$0 \xrightarrow{f} -2 \xrightarrow{f} 2 \circlearrowleft f$$

0 and -2 are eventually fixed points.

2 $\pm\sqrt{3}$ are also eventually fixed points

$$f(\pm\sqrt{3}) = (\pm\sqrt{3})^2 - 2 = 1$$

$$f(1) = (1)^2 - 2 = -1$$

$$+\sqrt{3} \xrightarrow{f} 1 \xrightarrow{f} -1 \circlearrowleft f$$

$$-\sqrt{3} \xrightarrow{f} 1 \xrightarrow{f} -1 \circlearrowleft$$

<u>Remark</u> here we are given these eventually fixed points:

$0, -2, +\sqrt{3}, -\sqrt{3}, 1$

and we checked they are ev. fix. pts.

If we had to find them you would have to solve

$f(x) = -1$ ← a fixed pt.

fixed pt.

$$x^2 - 2 = -1$$

$$x^2 - 1 = 0 \Rightarrow x = -1, +1$$

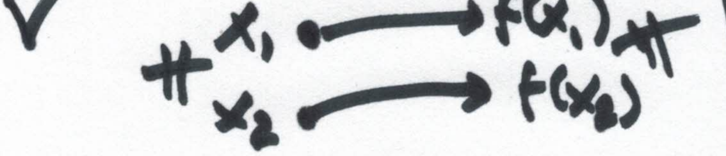
here we discovers the eventually fixed point $+1$.

<u>Homework</u> Find more eventually fixed points.

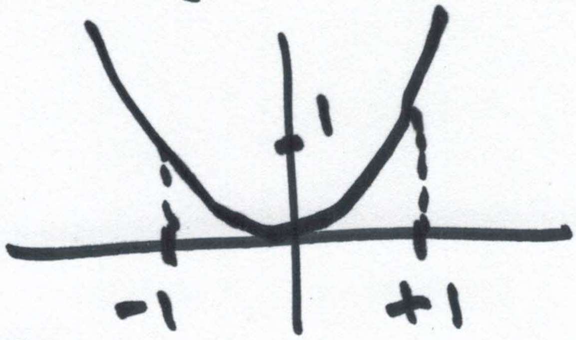
<u>Revision of mathematical terminology.</u>

<u>Definition</u> If $f : A \to B$ is a function and A, B sets then it is called one-to-one or injective if

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$

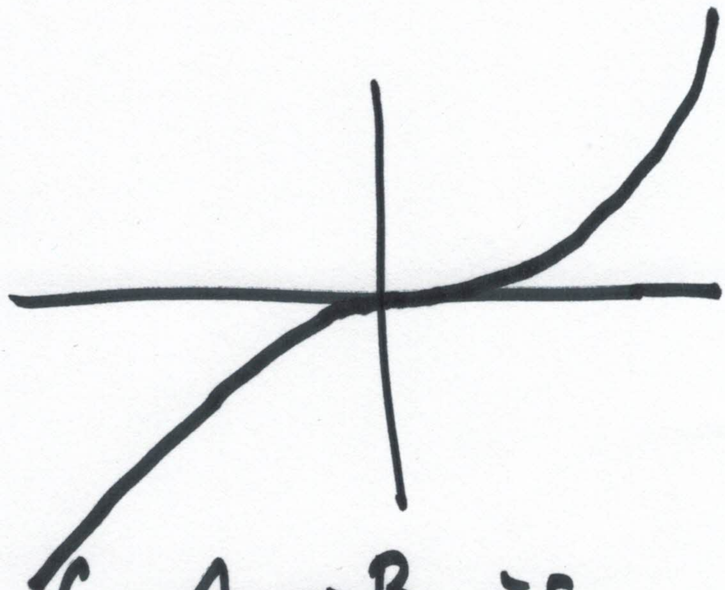


Examples 1. $f(x)=x^2$ is not injective as $f(-1)=f(+1)$



2. $f(x)=x^3$

this is injective.



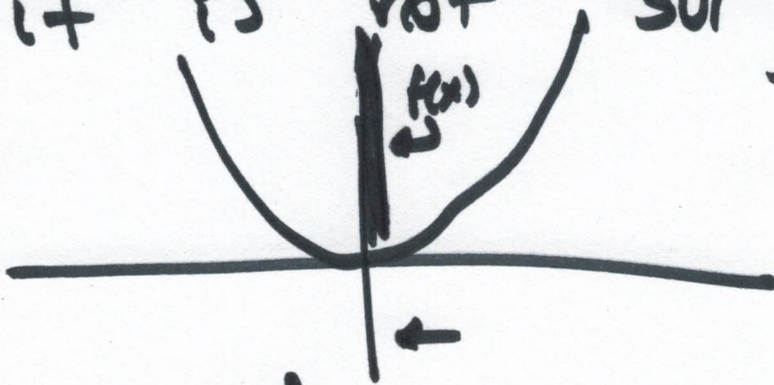
Definition If $f: A \to B$ is a function from the set A to the set B, then f is called onto or exhaustive (epijective) if:

exhaustive = surjective

for any $b \in B$ there exists some $a \in A$ s.t. $f(a)=b$

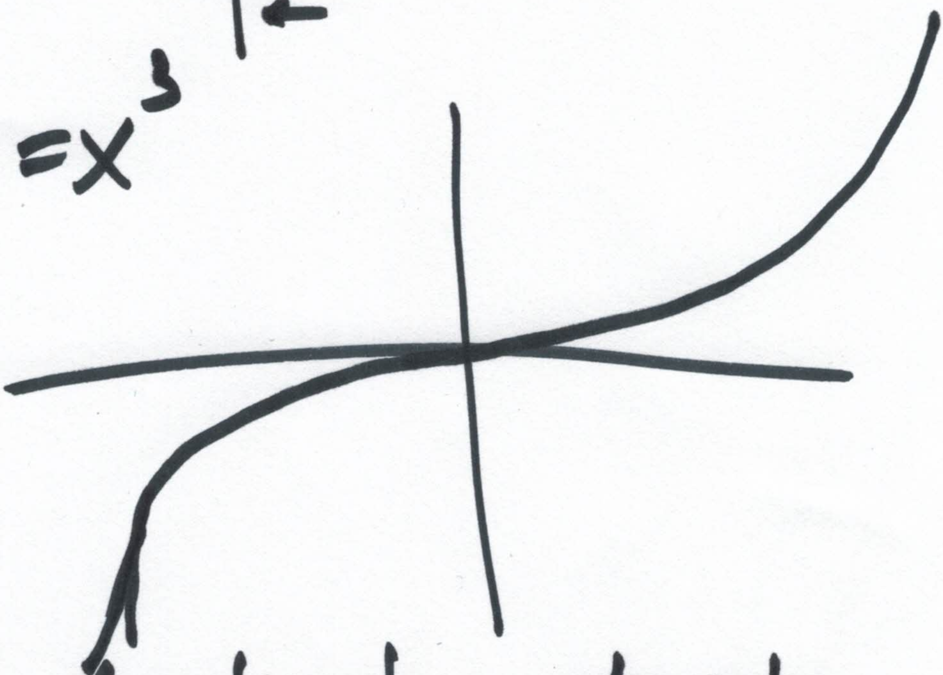
<u>Examples</u> ① $f(x) = x^2$

never yields negative numbers
So it is not surjective



② $f(x) = x^3$

is
surjective.



<u>Remark</u> In fact to decide whether something is injective or surjective you have to specify A & B, & answers can change with different A & B.

1. $f: \mathbb{R} \longrightarrow \mathbb{R}^+$ ← positive numbers

$x \longmapsto x^2$

$f(x) = x^2$ is surjective.

not injective as $f(1) = f(-1)$

2. $f: \mathbb{R}^+ \longrightarrow \mathbb{R}$

↖ positive numbers

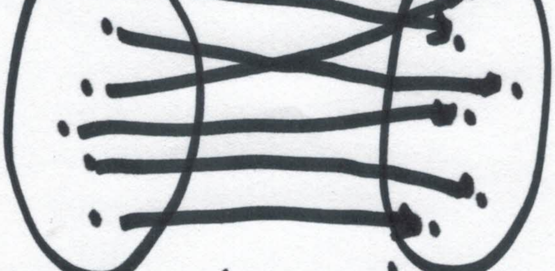
$f(x) = x^2$ not surjective

injective

Definition If $f: A \to B$ is both surjective and injective it is called bijective.

$$A \xrightarrow[f]{} B$$

"identification"

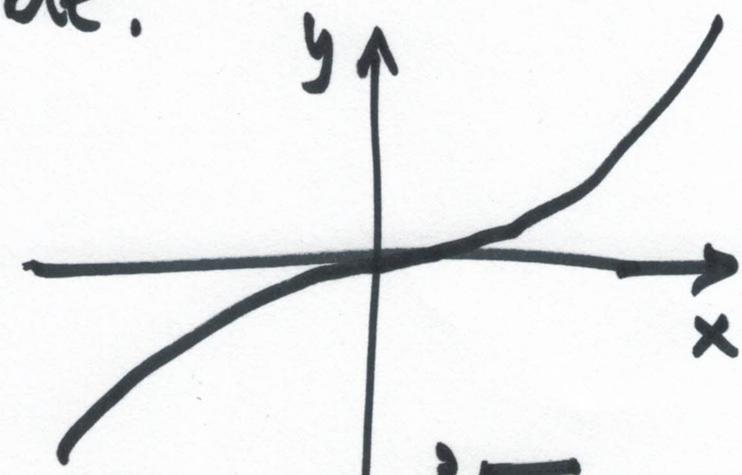


if f is bijective then we have an inverse function f^{-1}

Remark f bijective is also called invertible as then we have an inverse function f^{-1}

Examples 1 $f(x) = x^2$
$f : \mathbb{R} \to \mathbb{R}$
not invertible.

2 $f(x) = x^3$
$f : \mathbb{R} \to \mathbb{R}$
is bijective



with inverse $f^{-1}(x) = \sqrt[3]{x}$

there is unique cubic root for a real number (there are 3 in $\mathbb{C}$)

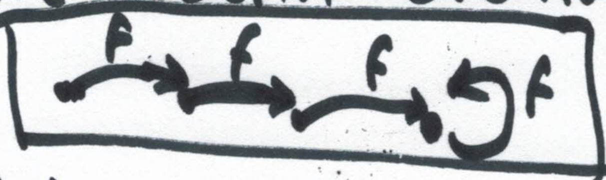
Homework Google injective/surjective bijective & make more examples.

Insert this @ the end of discussion on $f(x)=x^2-2$

We found fixed points $x^2-2=x$

$\hookrightarrow -1, 2$ $\boxed{\circlearrowleft^{f}}$

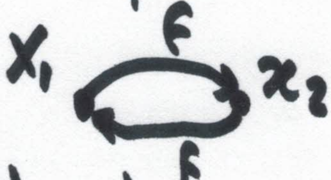
we found eventually fixed points



$1, \pm\sqrt{3}, 0$

Let us also see if there are periodic points

Let us check if there are points of period 2.



just need to solve

$$f(f(x))=x$$

$$f(x^2-2)=x$$

$$(x^2-2)^2-2=x$$

$$x^4 - 4x^2 - x + 2 = 0$$

But we know that the fixed points are also a solution to this as they also verify

$f(f(x)) = x$

Fixed points are -1 & 2

~~If you divide~~

To find the other 2 roots we divide polynomial by

$(x+1)(x-2)$

If you do this you get

$$x^2 + x - 1 = 0$$

The roots are

golden ratio.

$$\frac{-1 \pm \sqrt{5}}{2}$$

1.6...

0.6...

So then

$$\frac{-1+\sqrt{5}}{2} \underset{f}{\overset{f}{\rightleftarrows}} \frac{-1-\sqrt{5}}{2}$$

is a periodic orbit of period 2.

<u>Question</u> What to do to find period 3 orbits

$$\begin{tikzcd} \bullet \arrow[r, "f"] & \bullet \arrow[r, "f"] & \bullet \arrow[ll, bend left, "f"] \end{tikzcd}$$

<u>Answer</u> $f(f(f(x))) = x$

$$((x^2-2)^2-2)^2-2 = x$$

degree 6 polynomial eqn. ☹

but $x = -1, 2$ are solns.

$\Rightarrow$ can divide by $(x+1)(x-2)$

$\Rightarrow$ degree 4 polynomial eqn. ☹

Recall we have fixed pts & eventually fixed pts.

Lemma If $f : \mathbb{R} \to \mathbb{R}$ is injective then an eventually fixed point is in fact a fixed point.

Remark 1 In example $f(x) = x^2 - 1$ we had fixed pts $2, -1$ & eventually fixed pts $0, 1, \pm\sqrt{3}$

Lemma does not apply because $f(x)$ is not injective

2 Lemma will apply to $f(x) = -x^3$ as this is injective.

Here $-x^3 = x$

Eventually fixed = fixed $= \{0\}$

<u>proof</u>: We need to see that if $f^k(x_0)$ is fixed then x_0 is fixed.

as $f^k(x_0)$ is fixed:

$$f(f^k(x_0)) = f^k(x_0)$$

$$f^{k+1}(x_0) = f^k(x_0)$$

$$f^k(f(x_0)) = f^k(x_0)$$

$$f(f^{k-1}(f(x_0))) = f(f^{k-1}(x_0$$

as f injective

$$f^{k-1}(f(x_0)) = f^{k-1}(x_0)$$

same again

$$f^{k-2}(f(x_0)) = f^{k-2}(x_0)$$

$$\vdots$$

$$f(x_0) = x_0$$

$\Rightarrow x_0$ is fixed $\blacksquare$

<u>Remark</u> in plain language
if f is injective we can cross it out from both sides of an equation

$$\cancel{f}(\dots) = \cancel{f}(\dots)$$

In fact there is a more general statement.

<u>We have</u> - fixed pts 
- eventually fixed pts

<u>Also</u>
- periodic points 
- eventually periodic pts

<u>Lemma</u> If $f:\mathbb{R}\to\mathbb{R}$ is injective then any point that is eventually periodic is in fact periodic

<u>proof</u> = Homework.