Defn The <u>orbit</u> of $x_0 \in \mathbb{R}$ under $f : \mathbb{R} \to \mathbb{R}$ is the set

$$\mathcal{O}(x_0) = \{x_0, x_1, x_2, x_3, \ldots\}$$
$$= \{x_0, f(x_0), f^2(x_0), f^3(x_0), \ldots\}$$
$$= \{f^n(x_0) : n \geq 0\}$$

ie. The orbit is a <u>set</u> (rather than a <u>sequence</u>)

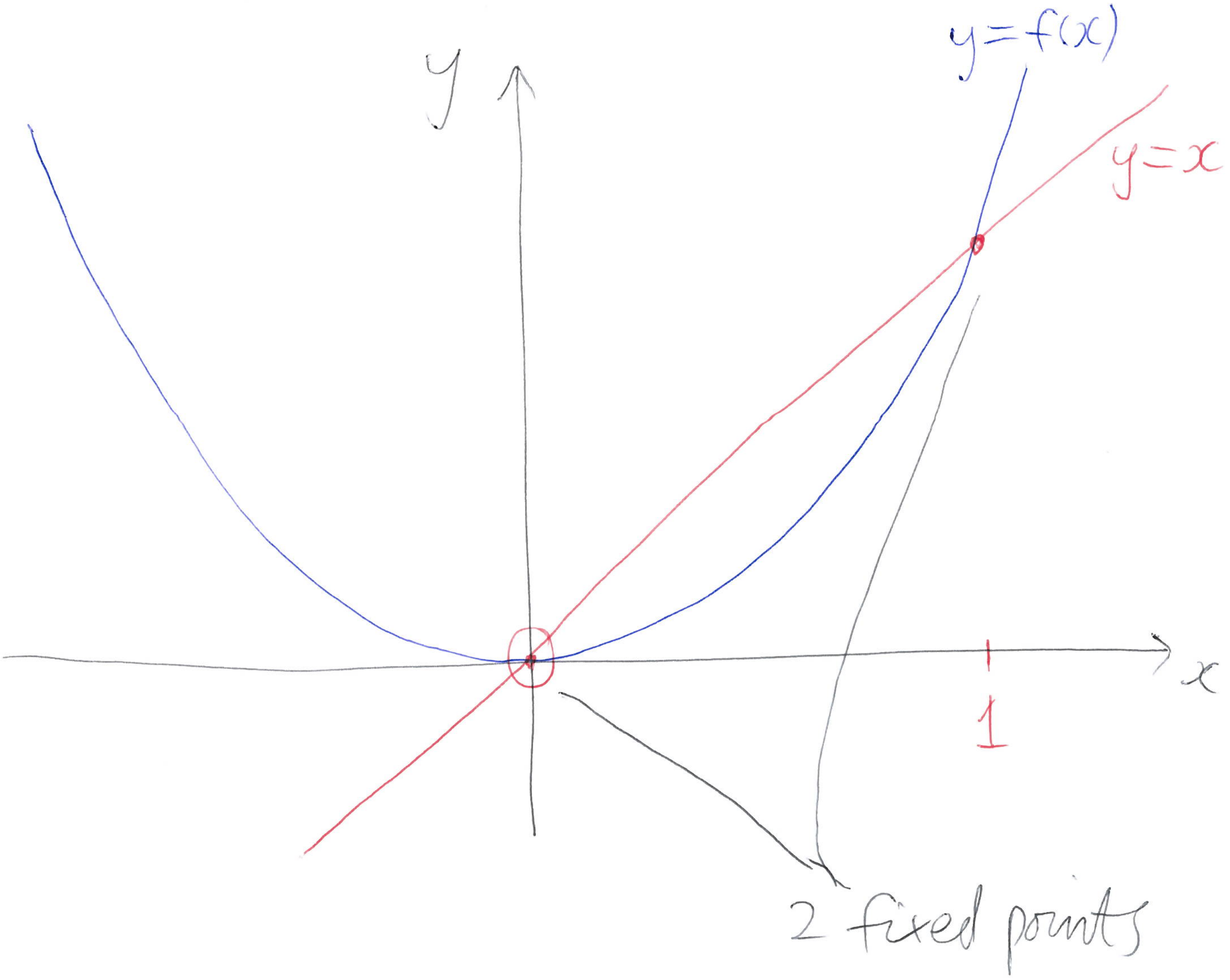
(though occasionally it is useful to blur the distinction, and use "orbit" to describe the <u>sequence</u> $(x_n)_{n=0}^{\infty}$)

Example Let us define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$.

First, recall the recurrence relation

$$x_{n+1} = f(x_n)$$

i.e. $x_{n+1} = x_n^2$



This function f has 2 fixed points, namely 0 and 1.

(These are the solutions to the equation $x^2 = x$, and clearly they are the only solutions, i.e. the only fixed points).

The orbit of any point is given by

$$\begin{array}{ccccccccc} x_0 & \xrightarrow{f} & x_1 & \xrightarrow{f} & x_2 & \xrightarrow{f} & x_3 & \xrightarrow{f} & \cdots \\ & & \| & & \| & & \| & & \\ & & x_0^2 & & x_1^2 & & x_2^2 & & \\ & & & & \| & & \| & & \\ & & & & x_0^4 & & x_1^4 & & \\ & & & & & & \| & & \\ & & & & & & x_0^8 & & \end{array}$$

Observe that if $|x_0| > 1$ then the values of $x_n = f^n(x_0) = x_0^{2^n}$ will get large (in fact they tend to ∞ as $n \to \infty$)

e.g. $x_0 = 3$

$$3 \xrightarrow{f} 9 \xrightarrow{f} 81 \xrightarrow{f} \dots$$

On the other hand if $|x_0| < 1$ then the values of $x_n = f^n(x_0)$ will get smaller, in fact converge to 0 as $n \to \infty$:

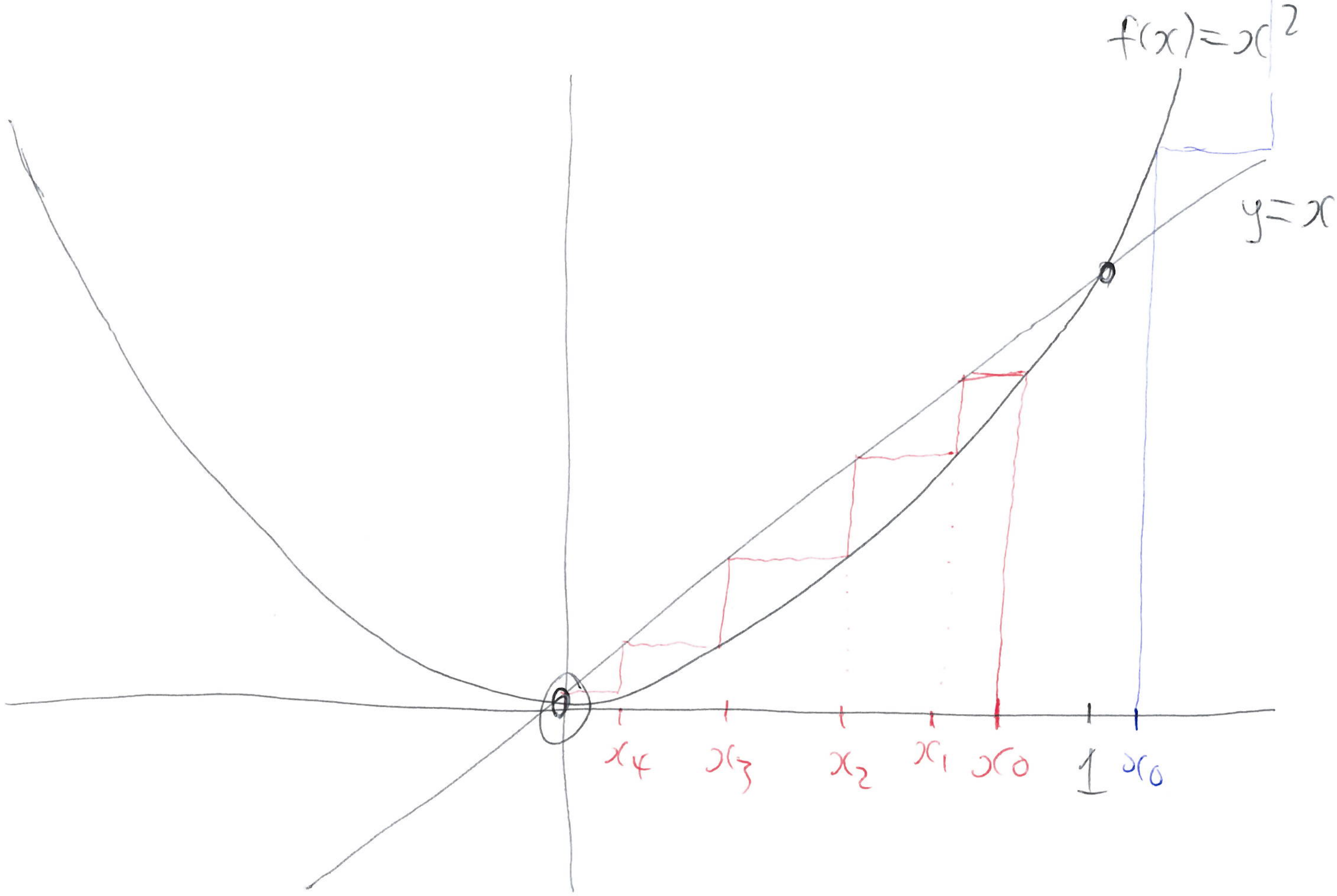
e.g. if $x_0 = 1/10 = 0.1$:

$$0.1 \xrightarrow{f} 0.01 \xrightarrow{f} 0.0001 \xrightarrow{f} \dots$$

Summary

$$\lim_{n\to\infty} f^n(x_0) = \begin{cases} 0 & \text{if } |x_0| < 1 \\ \infty & \text{if } |x_0| > 1 \\ 1 & \text{if } |x_0| = 1 \end{cases}$$

Returning to the graph of $f(x) = x^2$, we can depict an orbit using a so-called <u>cobweb diagram</u>



Notice that the fixed point 0 is "attracting", in the sense that if $|x_0| < 1$ then the sequence $x_0, x_1, x_2, \ldots$ ~~tends~~ converges to the fixed point.

By contrast, the fixed point 1 is "repelling".

<u>Question</u> Does this function $f(x) = x^2$ have any periodic points (apart from the two fixed points at 0 and 1)?

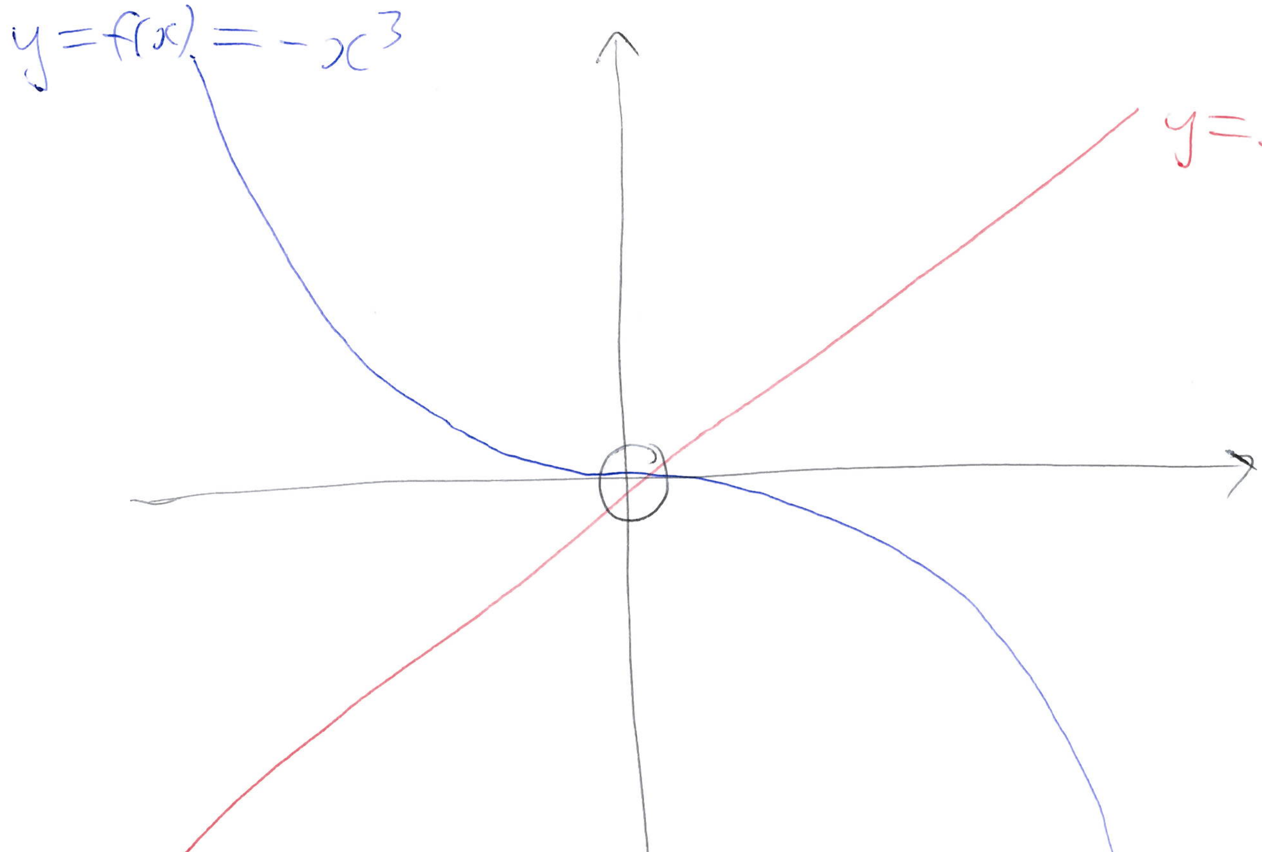
<u>Example</u> Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = -x^3$$

Examine fixed points and various orbits for this dynamical system f.

Recall the dynamical rule is given by

$$x_{n+1} = f(x_n) = -x_n^3$$



Fixed points: We need to solve the equation $f(x) = x$, i.e. $-x^3 = x$

i.e. $x^3 + x = 0$

i.e. $x(x^2+1) = 0$

The only (real) solution is $x = 0$.
i.e. 0 is the only fixed point

What about other periodic points?

Let's examine the orbit of any point x_0:

$$x_0 \xrightarrow{f} -x_0^3 \xrightarrow{f} x_0^9 \xrightarrow{f} -x_0^{27} \xrightarrow{f} \cdots$$

Notice that since the power ~~of~~ goes up, the 'size' of x_n is either growing or shrinking unless $x_0 = 1$ or $x_0 = -1$. So the only candidates for periodic points are $x_0 = 1$ and $x_0 = -1$.

Let $x_0 = 1$. Then the orbit of x_0 under f is

$$1 \xrightarrow{f} -1 \xrightarrow{f} 1 \xrightarrow{f} -1 \xrightarrow{f} \cdots$$

i.e.

$$\begin{tikzcd} 1 \arrow[r, bend left, "f"] & -1 \arrow[l, bend left, "f"] \end{tikzcd}$$

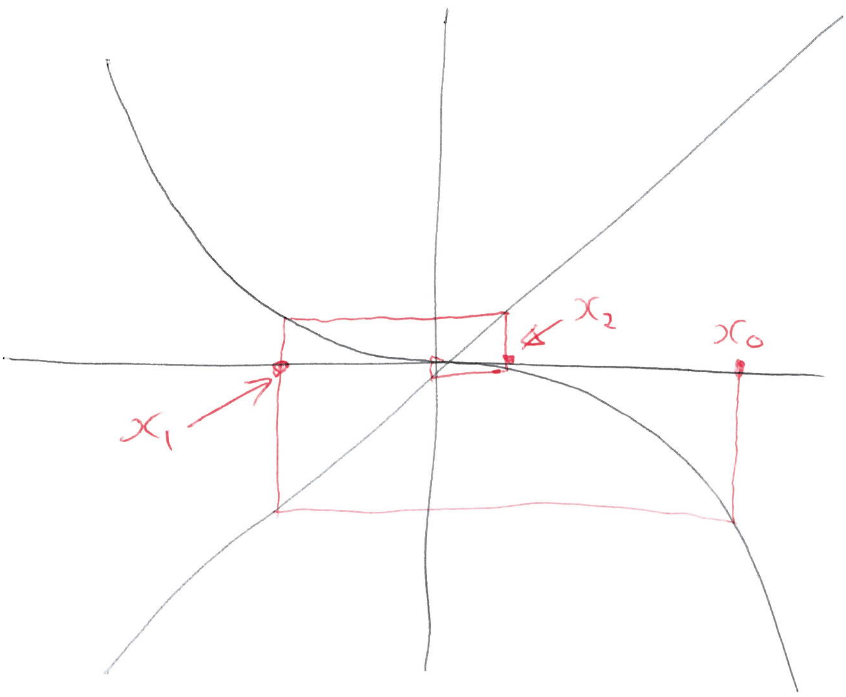
Thus 1 is a point of period 2 (i.e. a period-2 point), and similarly -1 is a period-2 point.

What about other $x_0 \in \mathbb{R}$?

We notice that $|f^n(x_0)| \to 0$ as $n \to \infty$ if $|x_0| < 1$, and $|f^n(x_0)| \to \infty$ as $n \to \infty$

if $|x_0| > 1$.

Thus 0 is an "attracting" fixed point



<u>Definition</u> Given a periodic point x_0, let m be the smallest natural number (i.e. smallest strictly positive integer) such that $f^m(x_0) = x_0$. Then m is called the <u>least period</u> (or <u>prime period</u>) of x_0.

~~Df~~

<u>Example</u> If $f(x) = -x^3$, then 0 is a fixed point, but it is also a point of period 2, and also a point of period 3, ... In fact a fixed point is a period-n point for <u>all</u> $n \geq 1$. The <u>least</u> period of this fixed point (in fact of <u>any</u> fixed point of <u>any</u> function f) is 1.

The point 1 is a period-2 point, but it is also a period-4 point, a period-6 point, etc. Its <u>least</u> period is 2.

<u>Definition</u> If x_0 has least period m, then the orbit $\{x_0, f(x_0), \ldots, f^{m-1}(x_0)\}$ is often called an <u>m-cycle</u> (or simply a <u>cycle</u>).

Question If x_0 has least period 2, can it also have period 7?

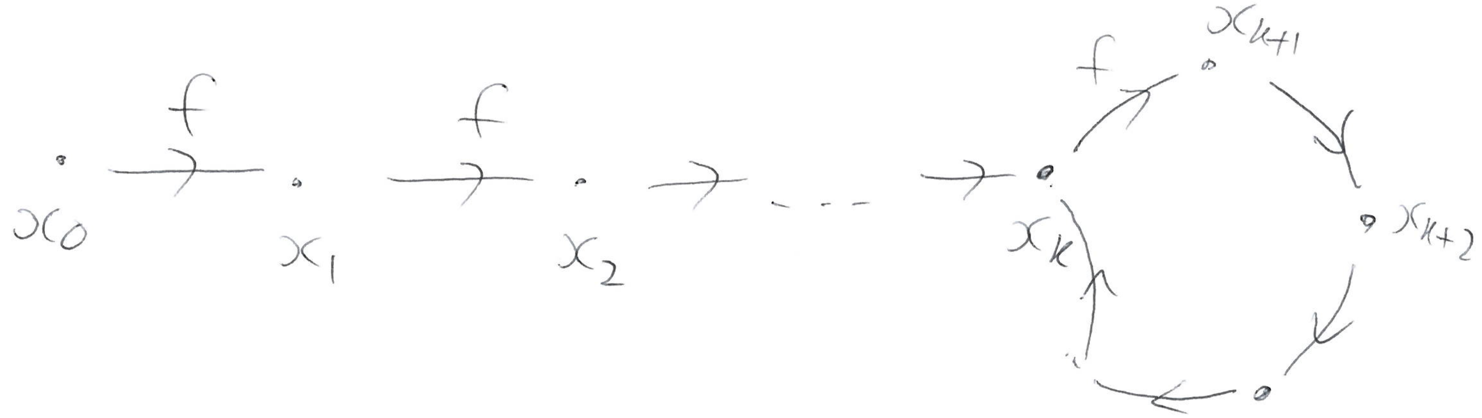
Lemma Let $x \in \mathrm{Per}_n(f)$
$$= \{x_0 : f^n(x_0) = x_0\},$$
and suppose x has least period k.
Then n must be an integer multiple of k.
(i.e. k must be a factor of n
i.e. k must be a divisor of n)

Proof Omitted / exercise. $\square$

Defn We say that a point x_0 is eventually periodic (or pre-periodic) ~~if~~ of period n if there exists some integer $k \geqslant 0$ such that

$f^k(x_0) = x_k$ is periodic of period n

$$\left(\text{i.e. } x_{k+n} = x_k \atop \text{i.e. } f^n\left(f^k(x_0)\right) = f^k(x_0) \right)$$



<u>Example</u> For $f(x) = x^2$, the point -1 is eventually periodic (of period 1) (we could say it is an <u>eventually fixed point</u>) since $f(-1) = 1$, and 1 is a fixed point.

$$-1 \xrightarrow{f} +1$$

<u>Example</u> Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 - 2$.

<u>Question</u>: Does f have any fixed points? If so, what are they?

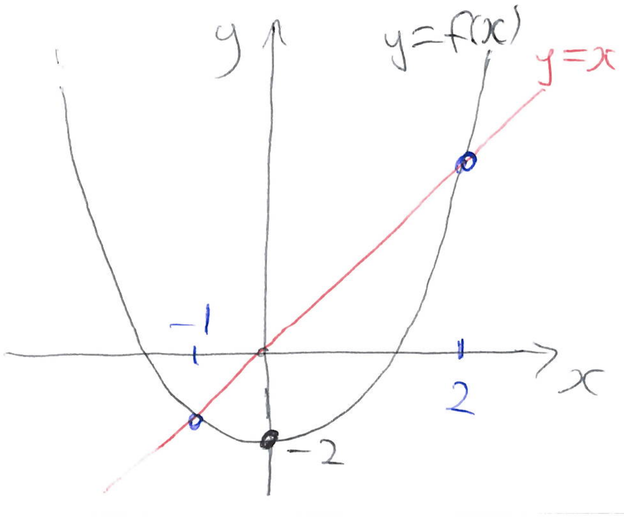
Fixed points are solutions of the equation $f(x) = x$

i.e. $x^2 - 2 = x$

i.e. $x^2 - x - 2 = 0$

i.e. $(x-2)(x+1) = 0$

i.e. $x = 2$ and $x = -1$ are the <u>only</u> fixed points of f



Note that the point -2 is an eventually fixed point, since

$$f(-2) = (-2)^2 - 2 = 4 - 2 = 2$$

Note also that 0 is an eventually fixed point, since

$$f(0) = 0^2 - 2 = -2.$$

We can represent this as:

$$\begin{tikzcd} 0 \arrow[r, "f"] & -2 \arrow[r, "f"] & 2 \arrow[loop right, "f"] \end{tikzcd}$$