Chaos + Fractals

Tuesday 10 – 12 PL-301

Wednesday 11 – 1 MB-204

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Assessment : 80% January exam (on campus)

: 20% Test (in-term)

'Chaos'

Dictionary definition:

'Complete disorder + confusion'

Maths / science definition:

The property of a complex system whose behaviour is so unpredictable as to appear random, owing to great sensitivity to small changes in conditions.

'Chaos Theory': The branch of maths that deals with complex systems whose behaviour is highly sensitive to slight changes in conditions, so that small alterations can give rise to strikingly great consequences.

Examples

Fluid flow: • Drop 2 nearby bottles from an ocean liner at sea, they will end up far apart.

- Drop 2 nearby sticks in "rapids" (fast flowing stream with rocks, etc), they will move apart before long

- Release 2 nearby (helium) balloons on a windy day, they will move far apart

- Weather prediction

Chaos + Fractals

- A 'new' branch of maths
  $\approx$ ~~50~~ 40–50 years old

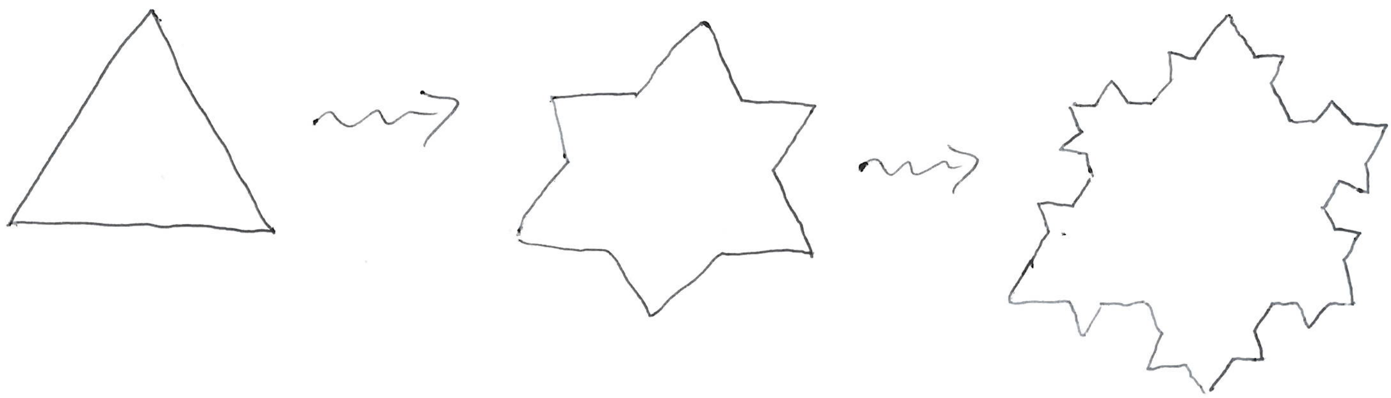
- Mathematicians working in Dynamical Systems started using computers in order to produce images of 'chaotic' phenomena

- Fractal usually means a set of non-integer dimension (the word was first used by Mandelbrot in 1975)

from "fractional"

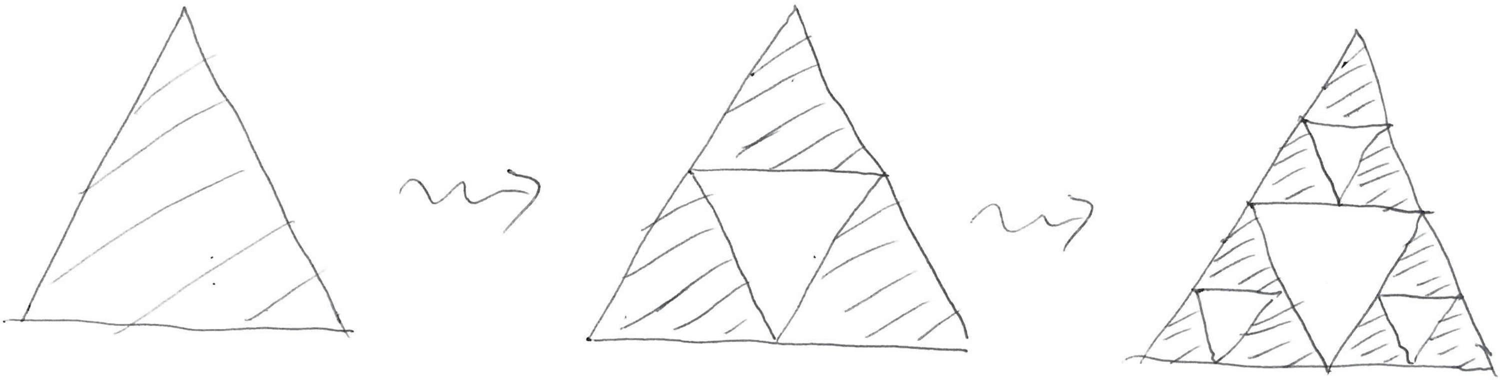
Examples of Fractals

(i) von Koch snowflake



⇝ ... ⇝ infinite process of refining the approximation to a geometric object which appears "in the limit" – this object turns out to have dimension > 1, and < 2 (i.e. dimension $\in (1, 2)$)

(ii) Sierpinski Triangle



⇝ ---

The "limit" is an object whose dimension is strictly between 1 and 2.

(iii) Snowflakes

(iv) Other 'fractals' in nature

- wiggly rivers
- ferns

- the coastline of Britain

Chapter 1 Dynamical systems (on $\mathbb{R}$)

Let $f: \mathbb{R} \to \mathbb{R}$.

This defines a recurrence relation:

$$\boxed{x_{n+1} = f(x_n) \quad \text{for all } n \geq 0}$$

which is a dynamical rule for generating a sequence $(x_n)_{n=0}^{\infty}$ provided we have a starting point $x_0 \in \mathbb{R}$.

This is because

$$x_1 = f(x_0)$$

$$x_2 = f(x_1) = f(f(x_0))$$
$$= f^2(x_0)$$

$$x_3 = f(x_2) = f(f(x_1))$$
$$= f(f(f(x_0)))$$
$$= f^3(x_0)$$

<u>Example</u> Take $f(x) = \cos(x)$

Choose $x_0 = 0 \cdot 8$

$x_1 = f(x_0) = 0 \cdot 6967\ldots$

$x_2 = f^2(x_0) = 0 \cdot 7669\ldots$

$x_3 = f^3(x_0) = 0 \cdot 7200\ldots$

$x_4 = f^4(x_0) = 0 \cdot 7517\ldots$

Continuing in this way, we observe that for large values of n,

$$x_n = f^n(x_0) \approx 0.739085\ldots$$

You can think of x_n as representing "position at time n".

In particular, x_0 is our "initial position", i.e. "position at time zero".

<u>Note</u> The dynamical rule (namely $x_{n+1} = f(x_n)\ \forall n \geq 0$) is <u>deterministic</u>.

Defn A point $x_0 \in \mathbb{R}$ is called a fixed point of f if

$$f(x_0) = x_0$$

For a given function f, define

$$\mathrm{Fix}(f) := \{ x_0 \in \mathbb{R} : f(x_0) = x_0 \}$$

to be the set of all fixed points of f.

A point $x_0 \in \mathbb{R}$ is said to be periodic (with period n) if

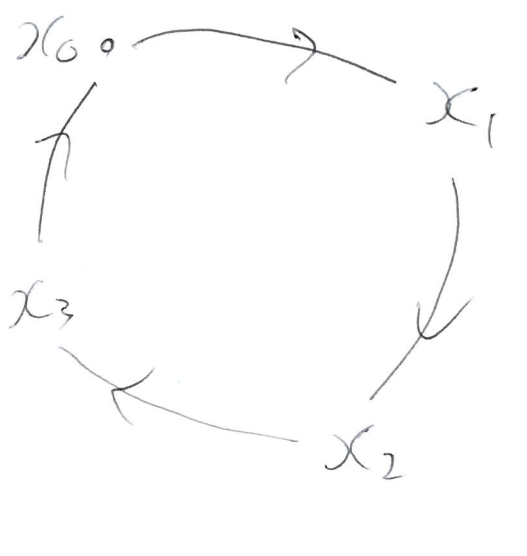
$$f^n(x_0) = x_0$$

i.e. $\underbrace{(f \circ f \circ f \circ \dots \circ f)}_{n \text{ times}}(x_0) = x_0$

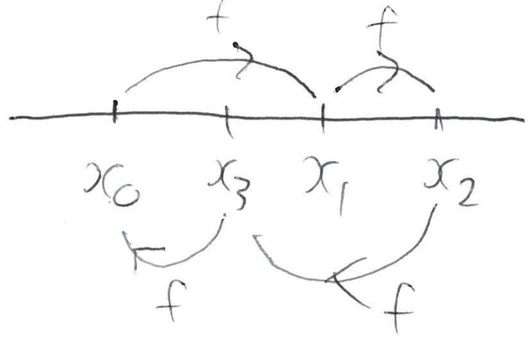
i.e. $\underbrace{f(f(f(\dots f(x_0)\dots)))}_{n \text{ times}} = x_0$

i.e. $x_n = x_0$ (Therefore $x_{n+1} = x_1$, $x_{n+2} = x_2$ etc)

Schematic picture illustrating period 4



Picture illustrating period 4



Warning Here $f^n(x_0)$ denotes the n-fold composition $\underbrace{(f \circ f \circ \dots \circ f)}_{n \text{ times}}(x_0)$

In particular, $f^n(x_0)$ does **NOT** mean that we "raise $f(x_0)$ to the nth power", i.e. does **NOT** mean $(f(x_0))^n$,

and $f^n(x_0)$ does **NOT** mean the nth derivative $f^{(n)}(x_0)$.

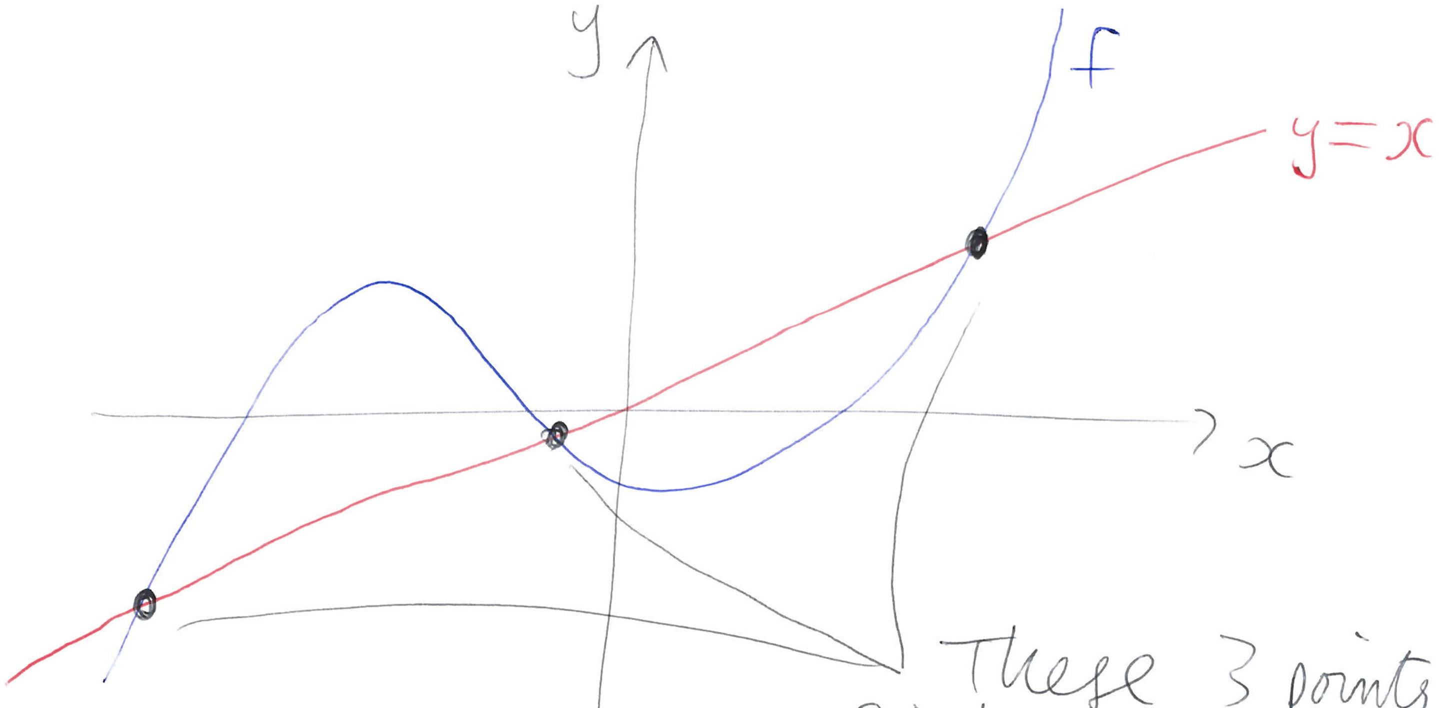
We denote the set of periodic points of period n by

$$\mathrm{Per}_n(f) = \{x_0 \in \mathbb{R} : f^n(x_0) = x_0\}$$

Note:

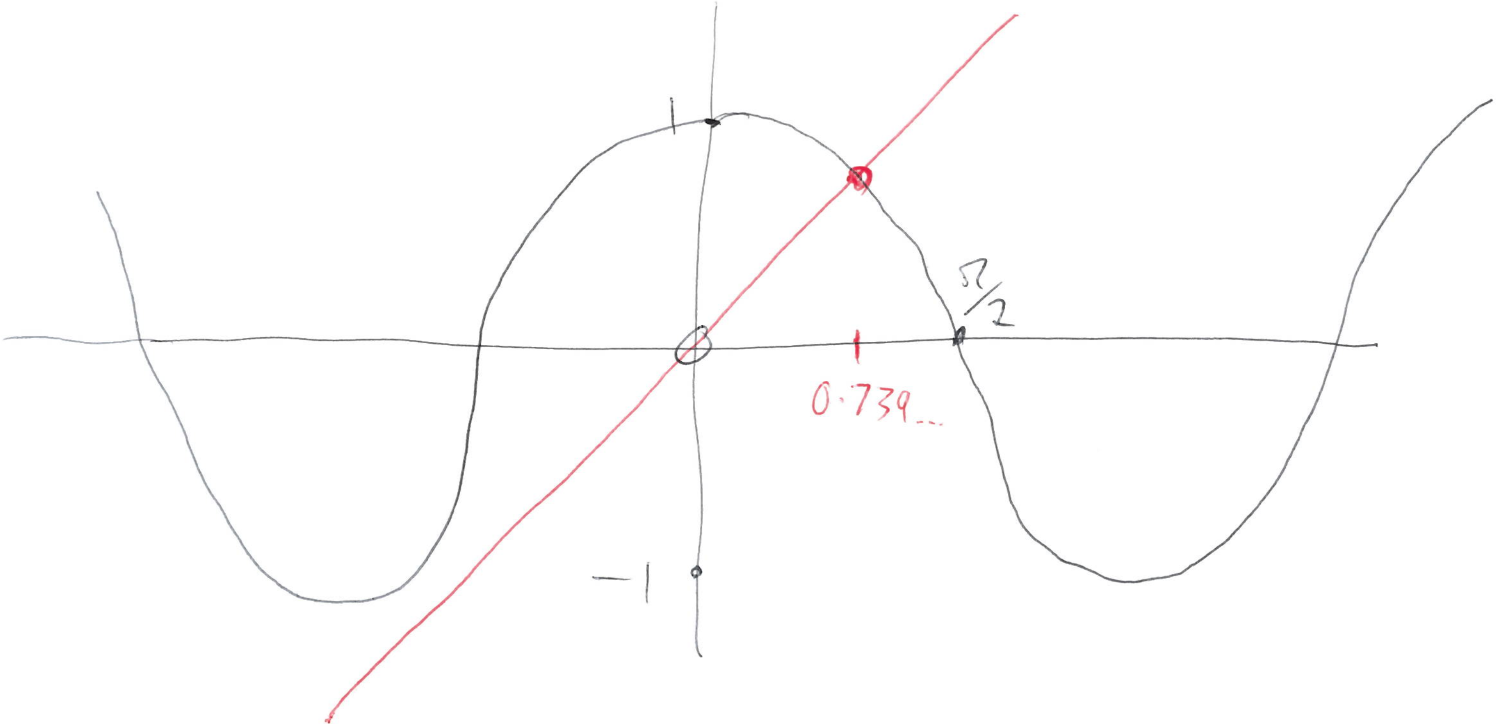
- $\mathrm{Fix}(f) = \mathrm{Per}_1(f)$
- $\mathrm{Fix}(f^n) = \mathrm{Per}_n(f)$
- $\mathrm{Fix}(f) \subset \mathrm{Per}_n(f) \ \forall n \geq 1$

Geometrically we can think of fixed points as intersections of the graph of f with the line $y = x$



These 3 points are fixed points of f i.e. satisfy $f(x_0) = x_0$

Example Recall we considered $f(x) = \cos(x)$



We previously "saw" that the cosine function has fixed point at 0.739085....