Review of some ODE techniques

$\boxed{\text{Method of integrating factors.}}$

$u(x)$ solves $\quad u'(x) + f(x)\cdot u(x) = 0$

Multiply ~~both~~ both sides by the integrating factor

$$e^{\int f(x)dx}$$

The equation becomes

$$e^{\int f(x)dx}\cdot u'(x) + e^{\int f(x)dx}\cdot f(x)\cdot u(x) = 0$$

Namely

$$\left[e^{\int f(x)dx}\cdot u(x)\right]' = 0$$

$$e^{\int f(x)dx}\cdot u(x) = C$$

$$u(x) = C\cdot e^{-\int f(x)dx}$$

Example 1: $u(x,y)$ is a solution to the PDE

$$u_x + x\cdot u = 0$$

Find $u(x,y)$.

The integrating factor is

$$e^{\int x\,dx} = e^{\frac{x^2}{2}}$$

Multiply both sides by the integrating factor,

get
$$e^{\frac{x^2}{2}} u_x + e^{\frac{x^2}{2}} \cdot x \cdot u = 0$$

Namely
$$\left[e^{\frac{x^2}{2}} u \right]_x = 0$$

Integrate both sides with respect to x, get

$$e^{\frac{x^2}{2}} u = f(y)$$

$$u(x,y) = f(y)\, e^{-\frac{x^2}{2}}$$

$\boxed{\text{Solving constant coefficient ODEs.}}$

u(x) solves

$$P_0 u(x) + P_1 u'(x) + P_2 u'' = 0$$

with P_0, P_1, P_2 constant

We consider the corresponding algebraic equation

$$P_0 + P_1 \cdot x + P_2 \cdot x^2 = 0 \qquad (*).$$

There are 2 cases:

Case 1: $(*)$ has 2 real roots

$$x_1 = a, \quad x_2 = b$$

then the general solution is

$$u(x) = c_1 e^{ax} + c_2 e^{bx} \quad \text{for any } c_1, c_2$$

Case 2: $(*)$ has a pair of complex roots

$$x_1 = a + ib$$

$$x_2 = a - ib$$

Then the general solution is

$$u(x) = c_1 e^{ax} \cos(bx) + c_2 e^{ax} \sin(bx)$$

for any c_1, c_2.

Example 2. Find the general solution $u(x,y)$ for the PDE

$$u_{yy} + u = 0$$

The corresponding algebraic equation is

$$x^2+1=0$$

which has a pair of complex roots

$$x_1 = i = 0 + 1 \cdot i$$

$$x_2 = -i = 0 - 1 \cdot i$$

we apply the above case 2 formula with

$a=0$, $b=1$, get

$$u(x,y) = f_1(x) e^{0 \cdot x} \cos(1 \cdot y) + f_2(x) e^{0 \cdot x} \sin(1 \cdot y)$$

$$= f_1(x) \cos y + f_2(x) \sin y$$

For any functions f_1, f_2.

(Notice here that as solutions to PDE, the constants c_1, c_2 may depend on x, so we use $f_1(x)$, $f_2(x)$ functions to denote them)