PROBLEM SET 1 FOR MTH 6151

1. Are the following partial differential equations (pde's) linear or nonlinear? Determine the order of the pde's as well.

(1) Transport equation $U_x + yU_y = 0$
(2) Shock wave $U_x + UU_y = 0$
(3) Laplace's equation $U_{xx} + U_{yy} = 0$
(4) Minimal surface equation $(1 + U_y^2)U_{xx} + (1 + U_y^2)U_{yy} - 2U_xU_yU_{xy} = 0$
(5) Mechanical wave $U_{xx} + U_{tt} = \sin U$
(6) Vibrating bar $U_{tt} + U_{xxxx} = 0$
(7) Mechanical wave $U_{xx} + U_{tt} = \sin U$

2. Which of the following operators are linear?

(1) $\mathcal{L}U = U_x + xU_y$
(2) $\mathcal{L}U = U_x + U_y^2$
(3) $\mathcal{L}U = \sqrt{1+x^2}(\cos y)U_x + U_{xyx} - \big(\arctan(x/y)\big)U$

3. For each equation, determine whether it is nonlinear, linear inhomogeneous, or linear homogeneous.

(1) $U_t + U_{xx} + 1 = 0$
(2) $U_t - U_{xx} + xU = 0$
(3) $U_x + e^yU_y = 0$
(4) $U_x(1 + U_x^2)^{-1/2} + U_y(1 + U_y^2)^{-1/2} = 0$

4. Show that the difference $V \equiv U_1 - U_2$ of two solutions U_1 and U_2 to an inhomogeneous linear pde $\mathcal{L}U = g$ (having the same g in both cases) gives a solution to the homogeneous pde $\mathcal{L}V = 0$.

5. Verify that $U(x, y) = f(x)g(y)$ is a solution of the pde

$$UU_{xy} = U_xU_y$$

for any differentiable functions f and g, of one variable.

6. Suppose $f(x)$ is differentiable and $c \neq 0$. Show $U(x, t) = f(x + ct)$ solves the equation

$$U_t - cU_x = 0.$$

7. Show that $U(x, t) = \text{sech}^2(x - t)$ solves the equation

$$4U_t + U_{xxx} + 12UU_x = 0.$$

Hint:

$$\frac{d}{dz}\text{sech}z = -\text{tanh}z\text{sech}z, \qquad \frac{d}{dz}\text{tanh}z = 1 - \text{tanh}^2 z.$$

8. Check that $U(x,t) = 4\arctan[e^{m(x-vt)/a}]$ is a solution of the equation

$$U_{xx} + U_{tt} = m^2 \sin U$$

for $a^2 = 1 + v^2$.