Powers and Logarithms

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How Logarithms are useful

Biology	Calculate Growth of bacteria or viruses using Log Machine	*Some bacteria grow double after 20 minutes, to check how much will it grow in 24 hours, biologist use log. Log is easy to interpret on calculator*
Archaeologist	Use logs and exponents to determine how old a fossil or artifact is	
Engineering	To calculate the magnitude of Earthquake	*The magnitude of an earthquakes is measured from the log of an amplitude of waves recorded by seismographs.*
Physics, Chemistry, Philosophy	Log used anywhere to manage long calculations	*Main purpose is to simplify the long calculations, we usually encounter long numbers which are not easy to divide or multiply, so we use log to short those numbers and then divide or multiply and then use ani-log to get the real numbers.*

Powers and Logarithms

What are the exponents: Exponents are repeated multiplication.

$$x^n = x \times x \times x \times x \ldots \times x \text{ (n- times)}$$

Exponents invlove two numbers

x - base, n Power, index, exponent

Exponents are higher integer values

$x^3,\ 3^6,\ 3^7$

Powers and Logarithms

$x^3 \cdot x^2 = x \cdot x \cdot x \cdot x \cdot x = x^5$

Exponential Law

1. $x^1 = x$
2. $x^m x^n = x^{m+n}$
3. $x^0 = 1$
4. $\dfrac{x^m}{x^n} = x^{m-n}$

Anything power 1 is itself

Any number raise to the power zero is 1

x^{-n} Repeated Division

as n is the counter part of n

Powers and Logarithms

$x^m \cdot x^n$ — Simplifying

Exponential Law

1. $x^{-n} = \dfrac{1}{x^n}$ $= \dfrac{1}{x \cdot x \cdot x \cdot \ldots x\ (n-\text{times})}$
2. $(xy)^m = x^m y^m$
3. $(x^m)^n = x^{mn}$
4. $\left(\dfrac{x}{y}\right)^n = \dfrac{x^n}{y^n}$

$\left(\dfrac{x}{y}\right)^3 = \dfrac{x^3}{y^3}$

$(xy)^3 = xy \cdot xy \cdot xy$

Commutative

$= x \cdot x \cdot x \cdot y \cdot y \cdot y$

$= x^3 \cdot y^3$

Distributing powers

Powers and Logarithms

Example

- What is $1^1 = ?$, $1^{100=?}$, $1^{-5} = ?$ $15p^3q^7$
- $\boxed{\text{Write } 5p^2q^3 \times 3pq^4 \text{ in its simplest form.}}$
- Express $\frac{1}{3^{-4x}}$ in the form of a^x for a suitable number a.

$$\frac{1}{3^{-4x}} = 3^{4x} = (3^4)^x = (81)^x$$

$$\boxed{a = 81}$$

Example: Simplify each expression given below

(i) $\sqrt[3]{8}$ $\qquad (2^3)^{1/3} = 2$

(ii) $\sqrt[3]{-125}$ $\qquad ((-5)^3)^{1/3} = -5$

(iii) $\sqrt[3]{40}$ $\qquad (40)^{1/3} = (2^3)^{1/3} \times \sqrt[3]{5}$

$= 2 \times \sqrt[3]{5}$

(iv) $(2\frac{3}{4})^5$?

(iv) $\left(\frac{11}{4}\right)^5 = \frac{11^5}{4^5} = \frac{161051}{1024}$

Exercise:

(i) 2^6

(ii) $8^{2/3}$

(iii) $\sqrt[3]{1000000}$ $= \left(10^6\right)^{1/3} = 10^2 = 100$

(iv) $(0.2)^{-2}$ $= \dfrac{1}{(0.2)^2} = \dfrac{1}{\left(\frac{1}{5}\right)^2} = 25$

Common Mistakes:

$$x^m + x^n \neq x^{m+n}$$

$$(x+y)^n \neq x^n + y^n$$

$$(x^m)^n \neq x^{m^n}$$

$$2^3 + 2^5 = 8 + 32 = 40$$

$$\neq$$

$$2^{(3+5)} = 2^8 = 256$$

$$(x^2)^3 = x^{2\times 3} = x^6$$

$$x^{2^3} \neq x^{2^3} = x^8$$

$$(2+3)^2 = 5^2 = 25$$

$$\neq$$

$$2^2 + 3^2 = 4 + 9 = 13$$

Powers and Logarithms

Simplify: $\dfrac{\sqrt[7]{a^{21}}}{(a^2a^3\sqrt{a})^4}$

$$= \frac{a^{21/7}}{a^8 a^{12} a^2}$$

$$= \frac{a^3}{a^{22}} = a^{-19}$$

Simplify: $\sqrt[3]{8}$

Powers and Logarithms

Exponentials: For exponentials we are interested where the base is fixed and the exponent is the variable. Fortunately calculations with exponentials follow the rules of powers as discussed above. $y = a^x$

Example

Solve the equations

(i) $10^{1-x} = 10^4$ (ii) $4^{5-9x} = \frac{1}{8^{x-2}}$

(iii) $10^{2x} = 10000$,

i) $1 - x = 4 \implies x = -3$

Powers and Logarithms

ii) $4^{5-9x} = 8^{-(x-2)}$

$5-9x = -x+2$

$\Rightarrow 5-2 = 8x \Rightarrow 8x = 3$

$x = \frac{3}{8}$

iii) $10^{2x} = 10^{4} \Rightarrow x = 2$

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Common Equations

Exponential Growth or Decay: $y(t) = ae^{kt}$
Where y(t)= value at time "t", a = value at the start, k= rate of growth (when $k > 0$) or decay (when $k < 0$), t=time.

Newton's law of cooling:

$$T(t) = Ae^{kt} + T_s$$

Logistic Growth:

$$f(x) = \frac{c}{1 + ae^{-bx}}$$

Powers and Logarithms

Definition: Logarithms are a way of looking differenty at exponents or indices or powers. In its simplest form, a logarithm answers the followings questions: **How many of one number do we multiply to get another number?**

Example: Consider the expression $\mathbf{2^5}$. This is an abbreviation of

$$\mathbf{2} \times \mathbf{2} \times \mathbf{2} \times \mathbf{2} \times \mathbf{2} \times \mathbf{2}$$

it seems its doubling something and keeps growing. This will make the equation $\mathbf{2^5} = \mathbf{32}$. **Now there are 3 no's we discuss**

(i) The number we multiply is called the "base/growth raet" and

(ii) "5 means how many times you will grow at that rate"

(iii) Now what is 32? Its the final result, it means you are 32 times bigger than you started.

Powers and Logarithms

Now we will see how this analogy of time and growing will help us:

Example As we seen already $2^5 = 32$, now lets see the following

$$2^0 = 1 \text{ (using index law)}$$

You have not grown at all, you are of same size as you started

$$2^{-1} = \frac{1}{2} \quad \text{(using index law)}$$

You have not grown at all.
You are of same size as you have started.

How to write it: $2^5 = 32 \iff \log_2(32) = 5$, we read this mathematical term as log base 2 of 32 equalls to 5.

$$2^0 = 1 \iff \log_2(1) = 0$$

$$2^{-1} = \frac{1}{2} \iff \log_2(\frac{1}{2}) = -1, \quad \text{yes lograthims can be negative numbers}$$

Powers and Logarithms

Example

What is $\log_3(81)$......?
I am no more at doubling machine, instead I am trippling , we are asking "how many 3s need to be multiplied together to get 81?"

$$3 \times 3 \times 3 \times 3 = 81, \text{ so we need 4 of the 3s}$$

Answer: $\log_3(81) = 4$.

Example: $\log_{\sqrt{3}}(27) = ?$

The question is, what power I must raise to get 27.
Lets try $\sqrt{3} \times \sqrt{3} \times \sqrt{3} \times \sqrt{3} = 3 \times 3 = 9$, this gives us 9, how many times do we need to get 27 this means we need $\sqrt{3}$, 2 more times,

$$\sqrt{3} \times \sqrt{3} \times \sqrt{3} \times \sqrt{3} \times \sqrt{3} \times \sqrt{3} = 27$$

How to write this in index form: Fractional indices

$$\left(3^{\frac{1}{2}}\right)^6 = 3^3 = 27$$

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Practice Questions

(i) $\log_5 25 =$

(ii) $\log_3 81 =$

(ii) $\log_2 32 =$

(iv) $\log_{10} 1000 =$

(v) $\log_4 1 =$

(vi) $\log_4(4) =$

(vii) $\log_2(\frac{1}{2} =$

(viii) $\log_3(\frac{1}{27}) =$

(ix) $\log_2(\frac{1}{16}) =$

(x) $\log_a(a^3) =$

(xi) $\log_4(-1) =$?

Negative

Why logorithms Argu cannot be

Powers and Logarithms

Answers

$$\log_5 25 = \mathbf{2}$$

$$\log_3 81 = \mathbf{4}$$

$$\log_2 32 = \mathbf{5}$$

$$\log_{10} 1000 = \mathbf{3}$$

$$\log_4 1 = \mathbf{0}$$

$$\log_4 4 = \mathbf{1}$$

$$\log_2\left(\frac{1}{2}\right) = -\mathbf{1}$$

$$\log_3\left(\frac{1}{27}\right) = -\mathbf{3}$$

$$\log_2\left(\frac{1}{16}\right) = -\mathbf{4}$$

$$\log_a(a^3) = \mathbf{3}$$

$$\log_4(-1) = \text{NOPE}$$

Powers and Logarithms

Logarithms Rules

No matter what a is

1. Identity Rule: $\log_a(a) = 1, \quad \log_a(1) = 0$ ✓
2. Product Rule: $\log_a(xy) = \log_a(x) + \log_a(y)$
3. Quotient Rule: $\log_a(\frac{x}{y}) = \log_a(x) - \log_a(y)$
4. Power Rule: $\log_a(x^n) = n\log_a(x)$
5. Change of Base Rule: $\boxed{\log_a(x) = \frac{\log_b x}{\log_b a}}$ very Imp Rule
6. Equality Rule: If $\log_a x = \log_a y$ then $x = y$.

When adding logs, multiply the argument.
When Subtracting logs, Devide the argument.

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Addition and Subtraction Rule

Example

(i) $5\log_8(2) + \frac{1}{2}\log_8(4)$

(ii) $\log_3(270) - (\log_3(2) + \log_3(5))$

(iii) $\log_a(a^2) + 3\log_a(a)$

i) $\log_8(2^5) + \log_8(4^{1/2})$

We have applied the power Rule.

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$= \log_8(32 \times 2)$ Product Rule

$= \log_8(64) = 2$

ii) $\log_3(270) - (\log_3(10))$

$= \log_3\left(\frac{270}{10}\right) = \log_3(27) = 3$

iii) $\log_a(a^2) + \log_a(a^3)$

$2 + 3 = 5$

Powers and Logarithms

Logarithms to Exponentials and Exponentials to Logarithms

The equation $a^x = c$ is equivalent to the equation $x = \log_a(c)$

Example

(i) $7^x = 49$

(ii) $3^x = 81$

(iii) $10^x = 10000$

(iv) $5^x = 0.2$

(v) $2^x = -8$

$2^x = (-2)^3$ bases are not same.

i) $x = 2$

ii) $x = 4$

iv) $5^x = \frac{1}{5} = 5^{-1}$

$x = -1$

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Rule of change of base for logarithms

V. Imp Rule

$$\boxed{\log_b(c) = \frac{\log_a c}{\log_a b}}$$

Example

Solve $4^x = 64$ using base $a = 4$ and $a = e$.

$$3 = \log_4(64) = x$$

$$\Rightarrow \quad x = 3$$

Powers and Logarithms

Lets change to base $a=2$

$$x = \frac{\log_2(64)}{\log_2(4)} = \frac{6}{2} = 3 \Rightarrow x=3$$

change to base $a=e$

$$x = \frac{\ln(64)}{\ln(4)} = 3$$

Example

Given that $\ln a = 2$ solve $a^x = e^6$ for x.

$$a^x = e^6$$

$$\ln(a^x) = \ln(e^6)$$

$$x \ln(a) = 6 \qquad \text{as } \ln(a) = 2$$

$$x = \frac{6}{2} = 3$$

Practice Question

Solve the simultaneous equation:

$$\log_2 y = \log_2 x + 4$$

$$2^{3y} = 8^y = 4^{2x+3} = 2^{4x+6}$$

$$\log_2 y = \log_2 x + 4\log_2(2)$$
$$= \log_2 x + \log_2(2^4)$$
$$= \log_2(16x) \implies y = 16x$$

Powers and Logarithms

Disguised Quadratics

$3y = 4x + 6$

$3 \times 16x = 4x + 6$

Example

Solve $e^{2x} + e^x = 2$ for x.

$\Rightarrow 48x - 4x = 6$

$44x = 6$

$x = \frac{6}{44}$

$x = \frac{3}{22}$

$y = e^x$

$y^2 + y - 2 = 0$

$y^2 + 2y - y - 2 = 0$

$y(y+2) - 1(y+2) = 0$

$(y-1)(y+2) = 0$

$y = 1$

$e^x = 1$

$x = \ln(1)$

$y = -2$

$e^x = -2 \longrightarrow$ Ignore

Powers and Logarithms

Practice Questions

1. $y^4 - 5y^2 - 36 = 0$
2. $q - 5\sqrt{a} - 36 = 0$
3. $2^{2x} - 2^{x+1} + 1 = 0$
4. $9(1 + 9^{x-1}) = 10 \times 3^x$

4. $9 + 9 \cdot 9^{x-1} = 10 \times 3^x$

$10 \times 3^x - 9^x - 9 = 0$

Powers and Logarithms

$$y = 3^x$$

$$y^2 = 3^{2x}$$

$$-9^x + 10\,3^x - 9 = 0$$

$$9^x - 10\,3^x + 9 = 0$$

$$y^2 - 10y + 9 = 0$$

$$y^2 - 9y - y + 9 = 0$$

$$y(y-9) - 1(y-9) = 0$$

$$y = 1, \quad y = 9$$

$$3^x = 1 = 3^0$$

$$x = 0$$

$$3^x = 3^2$$

$$\Rightarrow x = 2$$

Homework Worksheet 1

Exams Style Questions

Question 1: Solve for x and y the following system of simultaneous equations:

$$2\ln(x) = \ln(y) + \ln(5)$$

$$e^x e^y = e^3$$

Question 2: Find an expression for x given that $\ln(a) = 2$ and $a^x = e^6$.

Homework Worksheet 1
Exams Style Questions

$$\ln(x^2) = \ln(5y)$$

$$\Rightarrow x^2 = 5y \quad \text{—— } (1)$$

$x + y = 3$ —— (2)

$x = 3 - y$

Substitute the value x in (1)

$$(3-y)^2 = 5y$$

$$9 + y^2 - 6y = 5y$$

$$y^2 - 11y + 9 = 0$$

Quadratic formula

Homework Worksheet 1

Exams Style Questions

Question 3: Find the values of x satisfying

$$\log_2(x) + \log_2(x+5) = 2$$

Question 4 (2022): Solve the following logarithmic equation:

$$16\log_2 x + 4\log_4 x + 2\log_{16} x = 37, \quad x > 0.$$

Question 5 (2023): $2\log\left(\frac{x}{y}\right) - 1 = \log(10x^2y), \quad x \neq 0, y \neq 0$

Find the exact value of y.

Question 3: Hint $\log_2(x(x+5)) = \log_2(2^2)$

Homework Worksheet 1
Exams Style Questions

$$x^2+5x=4$$
$$x^2+5x-4=0$$

Question 4: Hint change of base law

$$\log_2(x^{16})+4\frac{\log_2(x)}{\log_2(4)}+2\frac{\log_2(x)}{\log_2(16)}=37\cdot\log_2(2)$$

$$\log_2(x^{16})+2\log_2(x)+\frac{1}{2}\log_2(x)=\log_2(2^{37})$$

$$\log_2(x^{16}\cdot x^2\cdot x^{1/2})=\log_2(2^{37})$$

Homework Worksheet 1

Exams Style Questions

$\Rightarrow x^{16+2+1/2} = 2^{37}$

Solve it further

Question 6: Why Can't Logarithms Argument Be Negative? Give Reason.

Question 7 (2023): Given that $a > 0, b > 0$ and $y > 0$, and that

$$2 + \log_a b + 3\log_a y = 2\log_a(a^2 y)$$

express y in terms of a and b, in form **not** involving logarithms.

Homework Worksheet 1
Exams Style Questions

Question 7: Hint:

$$2\log_a(a) + \log_a(b) + \log_a y^3 = 2\log(a^2 y)$$

$$\log_a(a^2) + \log_a(b) + \log_a(y^3) = \log(a^4 y^2)$$

$$\log_a(a^2 b \cdot y^3) = \log(a^4 y^2)$$

$$\Rightarrow \quad a^2 b y^3 = a^4 y^2$$

$$y = \frac{a^2}{b}$$

Activities for the Week 1

- Complete the Tasks of Week 1
- Attend your Lecture on Monday or Thursday
- Attempt End of Week 1 Quiz, Tutorial Sheet 1, sort out problems in Homework Worksheet 1
- Discuss any concerns at the Week 2 Feedback and support session on Friday or during the upcoming Lectures and Tutorials.
- Start Week 2 Tasks

End of Lecture