1. (a) Solve:

$$\log_2 \sqrt{x} + \log_4 x = 5.$$

[10]

(b) Find the two real values of x satisfying:

$$x - xe^{5x+2} = 0.$$

[15]

1a)

$$\log_2 \sqrt{x} + \log_4 x = 5$$

$$\log_2 x^{\frac{1}{2}} + \frac{\log_2 x}{\log_2 4} = 5$$

$$\frac{1}{2}\log_2 x + \frac{\log_2 x}{2} = 5$$

$$\frac{1}{2}\log_2 x + \frac{1}{2}\log_2 x = 5$$

$$\log_2 x = 5 \rightarrow x = 2^5 = 32$$

[both accepted]

1b)

$$x - xe^{5x+2} = 0$$

$$x(1 - e^{5x+2}) = 0$$

$$x = 0 \quad \text{or} \quad 1 = e^{5x+2}$$

$$e^0 = e^{5x+2}$$

$$\text{so} \quad 5x+2 = 0 \rightarrow x = -\frac{2}{5} = -0.4$$

[both accepted]

Two solutions : 0 and $-\frac{2}{5}$

2. (a) The sum of the remainders when $x^2 + 7x + (k - 20)$ is divided by $(x + 4)$ and $4x^2 + (k - 13)$ is divided by $(x - 3)$ is 15. Find the value of k.

[10]

(b) Separate into partial fractions:

$$\frac{4x^2 + 32x + 72}{(x + 5)^2(x - 1)}.$$

[15]

2a) $[x^2 + 7x + (k - 20)] / (x+4) \rightarrow r_1$

$[4x^2 + (k - 13)] / (x-3) \rightarrow r_2$

$r_1 + r_2 = 15$

the Remainder thm for the first division

$r_1 = (-4)^2 + 7 \cdot (-4) + (k - 20) =$

$= 16 - 28 + k - 20 = \underline{-32 + k}$

the Remainder thm for the second division

$r_2 = 4(3)^2 + k - 13 = 4 \cdot 9 + k - 13 =$

$= 36 - 13 + k = \underline{k + 23}$

adding the values obtained above and

$r_1 + r_2 = -32 + k + k + 23 = 2k - 9$

but $r_1 + r_2 = 15$ so equalling them to 15

$2k - 9 = 15$

$2k = 24 \rightarrow k = 12$

26)

$$\frac{4x^2+32x+72}{(x+5)^2(x-1)} = \frac{A}{x-1} + \underbrace{\frac{B}{x+5} + \frac{C}{(x+5)^2}}_{\text{[repeated factor rule]}} =$$

[linear factor

$$= \frac{A(x+5)^2 + B(x-1)(x+5) + C(x-1)}{(x-1)(x+5)^2}$$ ← common denominator

[equalling numerators]

$$4x^2+32x+72 = A(x+5)^2 + B(x-1)(x+5) + C(x-1)$$

at $x=1$ (root of the denominator)

$$4+32+72 = A(6)^2 \rightarrow 36A = 108 \rightarrow A=3$$

at $x=-5$ (root of the denominator)

$$4\cdot(-5)^2 + 32(-5) + 72 = C\cdot(-5-1) \rightarrow -6C = 12 \rightarrow C=-2$$

no more roots (any value allowed), e.g.

at $x=0$

$$72 = 5^2A + B(-1)(5) + C(-1)$$

$$72 = 25\cdot 3 - 5\cdot B + 2 \rightarrow -5B = -5 \rightarrow B=1$$

so $$\frac{4x^2+32x+72}{(x+5)^2(x-1)} = \frac{3}{x-1} + \frac{1}{x+5} - \frac{2}{(x+5)^2}$$

3. (a) Find the points with x coordinate $x = 2$ lying on the circle specified by the equation

$$(x-1)^2 + y^2 = 16.$$

And find the equations of the tangents at these points.

[12]

(b) Determine the centre and radius of the circle specified by the equation

$$x^2 + y^2 - 4x + 6y = 23.$$

Show that the point $P = (2, 3)$ lies on the circle. What is the equation of the radial line CP?

[13]

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3 a) $x = 2$ so

$(2-1)^2 + y^2 = 16$ ← plugging in the value of $x = 2$

$1^2 + y^2 = 16$

so $y^2 = 15 \rightarrow y = \pm\sqrt{15}$ [finding two sol. for y]

the two points are $(2, \sqrt{15})$ and $(2, -\sqrt{15})$

tangent at $P_1 = (2, \sqrt{15})$, $C = (1, 0)$ centre

[Note: notation might be different]

$$\text{grad } CP_1 = \frac{\sqrt{15} - 0}{2 - 1} = \sqrt{15}$$

$$\text{grad } t_1 = -\frac{1}{\sqrt{15}}$$

so t_1: $-\dfrac{1}{\sqrt{15}} = \dfrac{y - \sqrt{15}}{x - 2} \rightarrow y - \sqrt{15} = -\dfrac{1}{\sqrt{15}}(x-2)$

$$y = -\frac{1}{\sqrt{15}}x + \frac{2}{\sqrt{15}} + \sqrt{15} =$$

$$y = -\frac{\sqrt{15}}{15}x + \frac{2\sqrt{15} + 15\sqrt{15}}{15}$$

Equation of the tangent at P_1

[other accepted form: $y = -0.258x + 4.389$]

tangent at $P_2 = (2, -\sqrt{15})$

$$\text{grad } CP_2 = \frac{-\sqrt{15} - 0}{2 - 1} = -\sqrt{15}$$

$$\text{grad } t_2 = \frac{1}{\sqrt{15}} = \frac{\sqrt{15}}{15}$$

so t_2 : $\frac{1}{\sqrt{15}} = \frac{y + \sqrt{15}}{x - 2} \rightarrow y + \sqrt{15} = \frac{1}{\sqrt{15}}(x-2)$

$$y = \frac{1}{\sqrt{15}} x - \frac{2}{\sqrt{15}} - \sqrt{15}$$

$$= \frac{\sqrt{15}}{15} x - \frac{2\sqrt{15} + 15\sqrt{(15)}}{15}$$

[Equation of the tangent at P_2 ; can be left as

$$y = 0.258x - 4.389$$

3b) $x^2 + y^2 - 4x + 6y = 23$

$$(x-2)^2 + (y+3)^2 - 4 - 9 = 23$$

[completing the square for y]

$$(x-2)^2 + (y+3)^2 = 36$$

[completing square for x]

Centre: $C = (2, -3)$, $r = 6$

$P = (2, 3)$:

[substituting $y = 3$ into either of eqs]

$$LHS = 2^2 + 3^2 - 4 \cdot 2 + 6 \cdot 3 = 4 + 9 - 8 + 18 = 23 = RHS$$

[substituting $x = 2$ into either of eqs.]

so the point $(2, 3)$ lies on the circle

[confirming that the equation is satisfied]

CP line $x = 2$

[recognising that CP line is VERTICAL LINE with $x = 2$ equation]

4. (a) The distance of a point P from the line $4x - 3y = 1$ is equal 5 units. Derive an equation specifying the loci of all P with this property.

[10]

(b) Find the equation of the perpendicular bisector of AO line segment where O is the origin and $A = (2, 4)$.

[15]

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a) $P = (x_p, y_p)$

$4x - 3y = 1$ → distance of a point from the line

[implicit eq. of a line]

$4x - 3y - 1 = 0$

$a = 4$
$b = -3$
$c = -1$

[does not have to be written separately, as long as used correctly here]

$$d_p = \frac{ax_p + by_p + c}{\sqrt{a^2+b^2}}$$

[formula does not have to be written separately, can be directly used]

$$d_p = \frac{4 \cdot x_p + (-3)y_p - 1}{\sqrt{16+9}} = 5$$

[equalling perpendicular distance to 5]

$$\frac{4x - 3y - 1}{\sqrt{25}} = 5$$

$$4x - 3y - 1 = 25$$

$$4x - 3y - 26 = 0$$

Loci are described by the straight line

b)

$O = (0,0)$

$A = (2,4)$

$M = \left(\frac{0+2}{2}, \frac{0+4}{2}\right) = (1,2)$ midpoint

perpendicular bisector has gradient m which is perpendicular to grad AO

$\text{grad } AO = \frac{4-0}{2-0} = 2$

$m = -\frac{1}{2}$

perp. bisector goes through $M = (1,2)$

$-\frac{1}{2} = \frac{y-2}{x-1}$

$y - 2 = -\frac{1}{2}x + \frac{1}{2}$

$y = -\frac{1}{2}x + 2\frac{1}{2} = -\frac{1}{2}x + \frac{5}{2} =$

$= -0.5x + 2.5$

[all accepted]

5. Functions $f(x)$ and $g(x)$ are specified as

$$f(x) = \frac{1}{1+x} \qquad \text{and} \qquad g(x) = 2x^2 - 1.$$

(a) Specify the domain and range for $f(x)$ and $g(x)$.

[6]

(b) Specify if functions $f(x)$ and $g(x)$ are odd, even or neither.

[9]

(c) What is the domain and range so that $f(x)$ and $g(x)$ become invertible? Find the inverse functions $f^{-1}(x)$ and $g^{-1}(x)$ and specify the domain and range of the inverse functions.

[10]

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a) $f(x) = \frac{1}{1+x}$

$D = \mathbb{R} \setminus \{-1\} = \{x \neq -1\}$

$= (-\infty, -1) \cup (-1, +\infty)$

[accepted versions]

$R = \mathbb{R}$ or $R = \mathbb{R} \setminus \{0\}$

$= \{x \neq 0\}$ ← image

[both accepted]

$g(x) = 2x^2 - 1$

$D = \mathbb{R}$ [the only accepted answer]

$R = \mathbb{R}$ or $R = \{y \geq -1\} = I$

image

[both accepted]

b) $f(-x) = \frac{1}{1-x} = \frac{1}{-(-1+x)} = -\frac{1}{-1+x} \neq f(x)$

neither odd nor even

$g(-x) = 2(-x)^2 - 1 = 2x^2 - 1 = g(x)$ even

c) $f(x) = \frac{1}{1+x}$ is invertible (1-1, onto & total)

on $D = \{x \neq -1\} = \mathbb{R} \setminus \{-1\}$

and with $R = I = \mathbb{R} \setminus \{0\} = \{y \neq 0\}$

the inverse: $y = \frac{1}{1+x}$

$x = \frac{1}{1+y} \rightarrow 1+y = \frac{1}{x}$

$y = \frac{1}{x} - 1$

$f^{-1}(x) = \frac{1}{x} - 1$

$D_{f^{-1}} = \mathbb{R} \setminus \{0\}$ and $R_{f^{-1}} = \mathbb{R} \setminus \{-1\}$

domain of the inverse

range of the inverse

$g(x) = 2x^2 - 1$ is invertible (1-1, onto & total)

on $D = \{x \geq 0\}$ (or equivalently $D = \{x \leq 0\}$)

$= [0, \infty)$ [equiv. notation]

and with $R = I = \{y \geq -1\} = [-1, +\infty)$

image

the inverse

$y = 2x^2 - 1$

$x = 2y^2 - 1$

$x + 1 = 2y^2$

$\pm\sqrt{\frac{x+1}{2}} = y$ ← need to choose depending on the choice of D.

if $D = \{x \geq 0\} \Rightarrow f^{-1}(x) = \sqrt{\frac{x+1}{2}}$

if $D = \{x \leq 0\} \Rightarrow f^{-1}(x) = -\sqrt{\frac{x+1}{2}}$

taking $f^{-1}(x) = \sqrt{\frac{x+1}{2}}$, $D_{f^{-1}} = \{x \geq -1\}$

domain of the inverse

$R_{f^{-1}} = \{x \geq 0\}$

range of the inverse