Chapter 2

Polynomials

2.1 Quadratics

An expression of the form

$$ax^2 + bx + c, \qquad a, b, c \in \mathbb{R} \quad \text{and} \quad a \neq 0 \tag{2.1}$$

is called a **quadratic (polynomial)**.

If we assign a y-value to every x-value according to the specific form of the quadratic then we obtain a function $f(x) = ax^2 + bx + c$ or $y = ax^2 + bx + c$ which is called a **quadratic function**.

If we plot the values of $f(x)$ for different x-values we obtain a graph which is called a **quadratic curve**. An example is shown in figure 2.1.

Remark: We have seen that **quadratic (polynomials)**, **quadratic functions** and **quadratic curves** are strictly speaking three different concepts. However since they are closely related we typically refer them simply as **quadratics**; with the context determining which one we actually mean.

As seen in (2.1) a quadratic has three independent parameters a, b, c which are also known as **coefficients**. a is the coefficient of x^2 and must be non-zero - otherwise we get a linear function. b is the coefficient of x and c is the constant term. We will often think of c as the coefficient of $x^0 = 1$ as this viewpoint highlights that c is the coefficient of a power of x and hence no different to a and b.

Remark: Quadratics do not always appear in the exact form given in (2.1).
$x^2 - 1$ is a quadratic with no x-term as $b = 0$.
$4 - x^2 + 3x$ is a quadratic, despite the fact that the order of the terms differs from (2.1).

There is some quite general behaviour that we can easily read off a quadratic without the need for a detailed analysis.

> When $a > 0$ the quadratic curve opens to the top and has a **minimum**.
>
> When $a < 0$ the quadratic curve opens to the bottom and has a **maximum**.
>
> The value c gives the **y-intercept**, i.e. the value where the curve crosses the y-axis.

An immediate consequence is the following:

If a and c have opposite signs, i.e. we have either $a > 0$ and $c < 0$ or we have $a < 0$ and $c > 0$ then the quadratic curve must cross the x-axis in two distinct places, called **roots** or **zeros** as shown in figure 2.1 below. It is useful to know these 'facts' as they make our life easier.

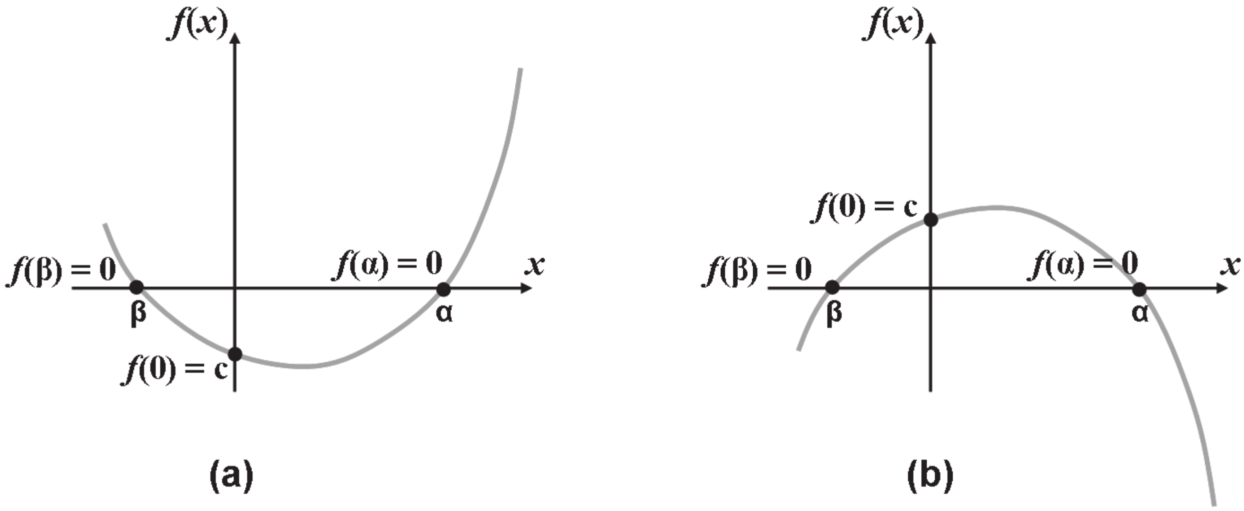

Figure 2.1: quadratic curves $f(x) = ax^2 + bx + c$ with (a) $a > 0, c < 0$ and (b) $a < 0, c > 0$.

2.2 The number of roots of a quadratic

Notice that both curves in figure 2.1 distinguish three points: the y-intercept c, where the curve crosses the vertical axis, and the values $x = \alpha$ and $x = \beta$, where $f(x)$ is zero and hence where the curve crosses the horizontal axis. Because $f(\alpha) = 0$ and $f(\beta) = 0$ the x-values α and β are called **zeros** of the quadratic function $f(x)$. Zeros are also commonly known as **roots**.

Note that a quadratic curve may not have any zeros: if the value of c in figure 2.1(a) were to lie above the x axis then the curve may not descend low enough to cross the x axis. The possible situations are sketched in figure 2.2

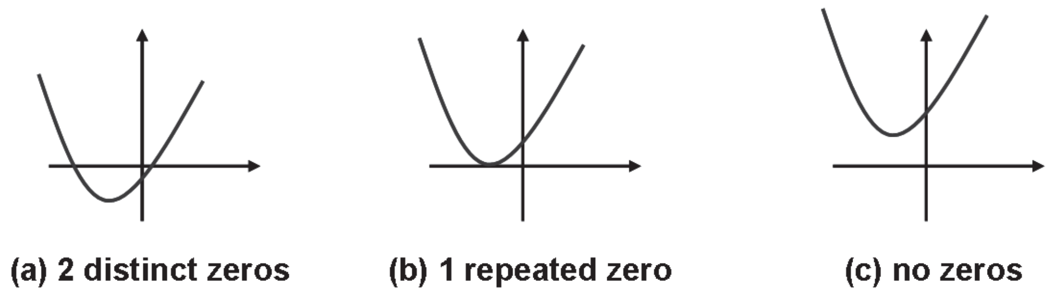

Figure 2.2: different possibilities for the zeros of a quadratic curve $f(x) = ax^2 + bx + c$, $a > 0$.

In order to find the roots of a given quadratic we need to solve the equation

$$ax^2 + bx + c = 0.$$

This equation has known explicit solutions x_1 and x_2 given by

$$x_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{2.2}$$

We often denote the roots x_1 and x_2 by α and β.

The term $b^2 - 4ac$ appearing under the root sign is known as the **discriminant** Δ. The sign of $\Delta = b^2 - 4ac$ determines how many roots a quadratic has:

When $\Delta > 0$ the quadratic has **two distinct roots**, Fig. 2.2(a).

When $\Delta = 0$ the quadratic has a **repeated root**, Fig. 2.2 (b).

When $\Delta < 0$ the quadratic has **no roots**, Fig. 2.2 (c).

Our earlier observations for $a > 0$ and $c < 0$, respectively $a < 0$ and $c > 0$, are easily explained since the term $-4ac$ is strictly positive and hence $\Delta > 0$.

Worked Example 2.1. Find the value(s) of k such that $x^2 + kx + 9$ has a repeated root.

Solution :

Worked Example 2.2. Show that for any real value of k the quadratic $x^2 + kx - 9$ can never have a repeated root.

Solution :

Symmetry axis of quadratic curves

The quadratic curve has an axis of symmetry located at $x = -\frac{b}{2a}$. This axis is of interest since the curve can be reflected back onto itself. More importantly the axis contains the maximum/minimum point, in other words the maximum/minimum value of a quadratic is located at $x = -\frac{b}{2a}$. It is hence easily found by evaluating $f(-\frac{b}{2a})$.

Moreover we can easily check that the axis lies exactly half-way between the two roots.

Exercise: Show that if α and β are the roots of (2.1) given by (2.2) then $\frac{\alpha+\beta}{2} = -\frac{b}{2a}$, i.e. the midpoint of the roots is located at $-\frac{b}{2a}$ as claimed.

Worked Example 2.3. Given the quadratic $f(x) = (2x + 3)(2x - 7)$ find the location and value of the minimum point.

Solution :

2.3 Factorisation of quadratics

The expression $(2x - 1)(x + 2)$ is the product of the two factors $(2x - 1)$ and $(x + 2)$. Multiplying out the brackets, $(2x - 1)(x + 2)$ gives rise to the quadratic $2x^2 + 3x - 2$. But what if we started with $2x^2 + 3x - 2$? How could we find the correct way to write this quadratic as a product of brackets of the form $(Ax + B)$? This process is known as **factorisation**.

The discriminant Δ helps us distinguish three cases

1. If $\Delta < 0$ then the quadratic has **no roots** and hence cannot be factorised.

   We call such a quadratic **irreducible**.

2. If $\Delta > 0$ then the quadratic has **two distinct roots** α and β given by (2.2) and

$$f(x) = a(x - \alpha)(x - \beta).$$

3. If $\Delta = 0$ then the quadratic has a **repeated root** $\alpha = \beta$ and

$$f(x) = a(x - \alpha)^2.$$

Remark: The formula (2.2) makes it easy to find α and β for the above factorisations. However it is important **not to forget the extra factor of a** in front of the brackets!

Remark: When one, or both, of the roots is a fraction then we may wish to factor out the denominator, e.g. $(x - \frac{3}{7}) = \frac{1}{7}(7x - 3)$.

Worked Example 2.4. Show that $f(x) = x^2 + 1$ and $g(x) = x^2 - 3x + 10$ are irreducible.

Solution :

Remark: Note that all irreducible quadratics can actually be factorised using complex numbers as their roots.

Worked Example 2.5. Factorise i) $f(x) = 3x^2 - 2x - 8$ ii) $g(x) = x + 2 - x^2$

Solution :

Worked Example 2.6. Find the quadratic function that has a repeated root at $x = 3$ and whose value at $x = 2$ is 5.

Solution :

Remark: We can save time if we can spot roots by inspection of the quadratic. We can quickly calculate $\Delta = b^2 - 4ac$ to see whether it is a square number; and thus whether there is any hope that the roots can be simplified.

Example: $x^2 - 7x + 5$ gives $\Delta = 29$, while $x^2 - 7x + 10$ gives $\Delta = 9$. Hence the former will involve terms with $\pm\sqrt{29}$, while in the second example we get a ± 3.

Quadratics with the same zeros

When we are given two zeros α and β as in figure 2.2(a) then there is an infinite family of quadratic curves $f(x) = ax^2 + bx + c$ which have these zeros, all of which are multiples of the other. The following example shows this more clearly:

The quadratic $f(x) = x^2 - 2x - 8 = (x + 2)(x - 4)$ has zeros $\alpha = -2$ and $\beta = 4$. As shown in figure 2.3 the functions $g(x) = -f(x) = -x^2 + 2x + 8$ and $h(x) = -4f(x) = -4x^2 + 8x + 32$ also have zeros at $\alpha = -2$ and $\beta = 4$. In fact any quadratic functions of the form $kf(x)$ for some k passes through $\alpha = -2$ and $\beta = 4$. Hence, as we pointed out there is an infinite family of such quadratic curves. This should be no surprise if we consider that we have 3 unknown coefficients in (2.1) to solve for but only two independent pieces of information given by the roots. This information is insufficient to fix the quadratic function and its quadratic curve uniquely.

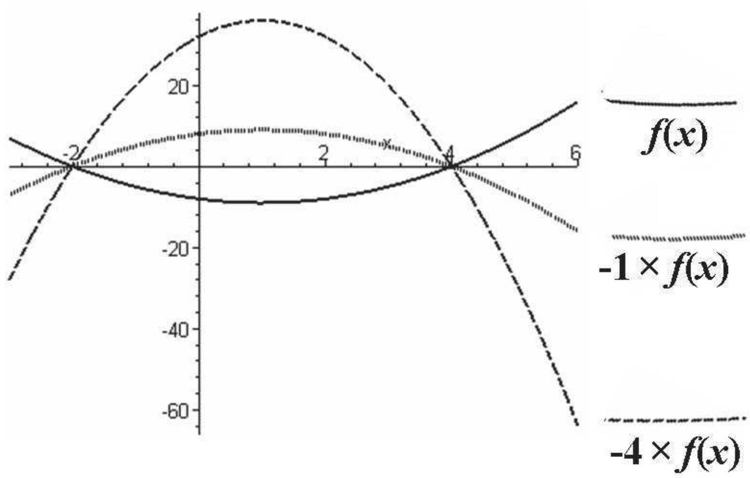

Figure 2.3: the quadratic curve $f(x) = x^2 - 2x - 8$ and two multiples of it, all having zeros at $x = 4$ and $x = -2$.

2.4 The method of equating coefficients

An important class of problems gives us a quadratic $f(x)$ with roots α and β and asks us to find a quadratic $g(x)$ whose roots γ and δ are related to α and β through some formula. For example we may be given the condition that $\gamma = \frac{1}{\alpha}$ and $\delta = \frac{1}{\beta}$.

The important part is that we want to find $g(x)$ without actually finding α and β explicitly. The reason is that if α and β involve surds $\sqrt{\Delta}$ then the expressions for γ and δ become tedious.

An important tool is the following method:

The method of equating coefficients

Two polynomials are identical as functions, that is they are identical for every single value of x, if and only if all the corresponding coefficients for the powers of x agree.

Let's take a look at the case when we use to quadratics, say $f(x) = ax^2 + bx + c$ and $g(x) = Px^2 + Qx + R$. We want $f(x) = g(x)$ for all values of x. We write this as $f(x) \equiv g(x)$

So according to our method $f(x) \equiv g(x)$ if and only if

$$\begin{aligned} x^2 :& \quad a = P \\ x^1 :& \quad b = Q \\ x^0 = 1 :& \quad c = R \end{aligned}$$

Worked Example 2.7. Suppose $ax^2 + 7x - 2 \equiv -x^2 + Qx + R$. Find a, Q, R.

Solution :

Worked Example 2.8. Suppose $7 - 3x^2 + kx \equiv px^2 - q$. Find k, p, q.

Solution :

If the quadratic $f(x) = ax^2 + bx + c$ has roots α and β then

$$\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a} \tag{2.3}$$

Worked Example 2.9. Derive equation (2.3).

Solution :

New roots from old

Method:

1. Determine the values of $\alpha + \beta$ and $\alpha\beta$ using (2.3).
2. Using the relationship that relates α and β to γ and δ find expressions for $\gamma + \delta$ and $\gamma\delta$ in terms of $\alpha + \beta$ and $\alpha\beta$.
3. Relate $\gamma + \delta$ and $\gamma\delta$ to the coefficients of $g(x)$ using (2.3).

Remark: We are free to set the x^2-coefficient equal to 1 in $g(x)$.

Worked Example 2.10. Let $f(x) = 2x^2 + 7x - 3$. Denote its roots by α and β.
Find a quadratic $g(x) = x^2 + px + q$ with $\gamma = \frac{1}{\alpha}$ and $\delta = \frac{1}{\beta}$.

Solution :

Worked Example 2.11. Let $f(x) = 2x^2 + 7x - 3$. Denote its roots by α and β. Find a quadratic $g(x) = x^2 + px + q$ with $\gamma = \beta + \frac{1}{\alpha}$ and $\delta = \alpha + \frac{1}{\beta}$.

Solution :

Remark: Observe that the roots of $f(x) = 2x^2 + 7x - 3$ are $x_{1/2} = \dfrac{-7 \pm \sqrt{73}}{4}$. The explicit expressions for $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ are tricky; and those for $\gamma = \beta + \frac{1}{\alpha}$ and for $\delta = \beta + \frac{1}{\alpha}$. However, in both cases $g(x)$ is very manageable.

Exercise: Given $f(x) = 2x^2 + 7x - 3$ find explicit expressions

1. for the roots α and β;
2. for $\dfrac{1}{\alpha}, \dfrac{1}{\beta}, \beta + \dfrac{1}{\alpha}$ and $\beta + \dfrac{1}{\alpha}$;
3. for the quadratic that has roots $\gamma = \dfrac{1}{\alpha}$ and $\delta = \frac{1}{\beta}$;
4. for the quadratic that has roots $\gamma = \beta + \dfrac{1}{\alpha}$ and $\delta = \alpha + \dfrac{1}{\beta}$.

2.5 Factorising polynomials

We have seen how to find the roots of a quadratic using (2.2) and thus factorising a quadratic explicitly. However for higher order polynomials formulas for the roots are either complicated or do not exit. A very important result in Mathematics states

The Fundamental Theorem of Algebra

Every polynomial with real coefficients can be broken down into a product of linear factors $(x - \alpha)$ and irreducible quadratics.

Remark: This result is not examinable, however it has some important implications for us:

1. If the highest power of x is odd then there is an odd number of (possibly repeated) roots, and hence there is at least one root.
2. If the highest power of x is even then there is an even number of (possibly repeated) roots.

There are two immediate questions for us

1. How can we possibly find roots without a formula?
2. Having found a root, how can we factorise the original polynomial?

The **Factorisation Theorem** below can help us with the first question, while **polynomial division** while help us address the second question.

If $(x-\alpha)$ is a factor of the polynomial $f(x)$ then by definition we can write $f(x)$ as $f(x) = (x-\alpha)g(x)$. Thus it follows that if $(x-\alpha)$ is a factor of the polynomial $f(x)$ then $f(\alpha) = 0$ as $f(\alpha) = (\alpha-\alpha)g(\alpha) = 0$. The reverse is true, too.

The Factorisation Theorem

Suppose we have $f(\alpha) = 0$. Then by definition α is a root and $(x - \alpha)$ is a factor of $f(x)$.

Note that the theorem does not offer a formula for finding roots, but we can use it if we have a lucky/educated guess.

Worked Example 2.12. Show that $x = 1$, $x = 2$ and $x = 3$ are roots of $f(x) = x^3 - 6x^2 + 11x - 6$ and hence factorise $f(x)$.

Solution :

Worked Example 2.13. [Adapted from Maths diagnostic test] Show that none of $x = 1$, $x = 2$ and $x = 3$ is a root of $f(x) = x^3 - 6x^2 + 11x + 6$ and hence none of $(x - 1)$, $(x - 2)$ and $(x - 3)$ is a factor of $f(x)$.

Solution :

2.6 Polynomial division and the Remainder Theorem

If we know that $f(x) = (x - \alpha)g(x)$ then we want to know how to find $g(x) = f(x)/(x - \alpha)$. Similarly we may wish to simplify expressions like

$$1)\quad \frac{2x^3 + x^2 + 4}{x - 1} \quad \text{or} \quad 2)\quad \frac{x^2 - 3x - 2}{x + 2}$$

Let's start with a quick review of long divisions with numbers.

Suppose we are asked to express the fraction $\frac{2917}{13}$ as mixed fraction $A\frac{b}{c}$. It is clear that $c = 13$. But how many whole units A do we get and what is the remainder b?

Worked Example 2.14. Using long division show that $\frac{2917}{13} = 224\frac{5}{13}$.

Solution :

It is good to spell out our steps in detail:

1. List multiples of 13
2. Focus on the first two digits and subtract a suitable multiple of 13 just below.
3. Take the remainder and combine it with the next digit.
4. Repeat steps 2 and 3 until there are no digits left over
5. The number left over is the remainder.

Now suppose we are asked to do the same with polynomials $f(x)$ and $g(x)$.

$$\frac{f(x)}{g(x)} = \text{quotient} + \frac{\text{remainder}}{g(x)} \tag{2.4}$$

$$\text{or} \quad f(x) = \text{quotient} \times g(x) + \text{remainder} \tag{2.5}$$

where the *quotient* and the *remainder* are also polynomials.

For polynomial division the steps are as follows:

Polynomial division

1. List multiples of $g(x)$
2. Focus on the highest power and subtract a suitable multiple of $g(x)$ just below so that highest power is eliminated.
3. Take the remainder and combine it with the next power (even if is missing!).
4. Repeat steps 2 and 3 until there are no power left over, i.e. you have used the constant
5. The term left over is the remainder.

Worked Example 2.15. Divide the cubic polynomial $2x^3 + x^2 + 4$ by $x - 1$.

Solution :

Here $f(x) = 2x^3 + x^2 + 4$ and $g(x) = x - 1$. So we need to repeatedly multiply $x - 1$ by powers of x and subtract them from the last line. We must be careful to include even those powers of x which are missing (i.e. have coefficient zero):

$$\begin{array}{rc|ccccccccl}
 & & 2x^2 & + & 3x & + & 3 & & & \longleftarrow & \text{quotient} \\
\hline
 & (x-1) & 2x^3 & + & x^2 & + & 0x & + & 4 & & \\
2x^2 \times (x-1) & \longrightarrow & 2x^3 & - & 2x^2 & & & & & & \\
\text{subtract:} & & & & 3x^2 & & & & & & \\
3x \times (x-1) & \longrightarrow & & & 3x^2 & - & 3x & & & & \\
\text{subtract:} & & & & & & 3x & & & & \\
3 \times (x-1) & \longrightarrow & & & & & 3x & - & 3 & & \\
\text{subtract:} & & & & & & & & \underline{7} & \longleftarrow & \text{remainder}
\end{array}$$

So

$$\begin{aligned}
\frac{2x^3 + x^2 + 4}{x - 1} &= \frac{(2x^2 + 3x + 3) \times (x - 1) + 7}{x - 1} \\
&= \frac{(2x^2 + 3x + 3) \times (x - 1)}{x - 1} + \frac{7}{x - 1} \\
&= 2x^2 + 3x + 3 + \frac{7}{x - 1}.
\end{aligned}$$

We see that the quotient is $q(x) = 2x^2 + 3x + 3$ and the remainder polynomial is $R(x) = 7$.

Worked Example 2.16. Use long division to simplify $\dfrac{x^2-3x-2}{x+2}$ and find the remainder.

Solution :

We pointed out earlier that we may write

$$\frac{f(x)}{g(x)} = q(x) + \frac{R(x)}{g(x)}$$

as

$$f(x) = q(x)g(x) + R(x).$$

Now suppose α is a root of $g(x)$ then substituting $x = \alpha$ into the last equation gives us

$$f(\alpha) = q(\alpha) \times 0 + R(\alpha) = R(\alpha).$$

So $f(\alpha) = R(\alpha)$. This is known as:

The Remainder Theorem

If $x = \alpha$ is a root of $g(x)$, i.e. $g(\alpha) = 0$, then we have $f(\alpha) = R(\alpha)$.

Worked Example 2.17. Find the remainders of i) $\dfrac{2x^3+x^2+4}{x-1}$ and ii) $\dfrac{x^2-3x-2}{x+2}$ using the Remainder Theorem and compare them with the results from the long divisions.

Solution :

Worked Example 2.18. Let $f(x) = x^6 - 103x^5 + 396x^4 + 3x^2 - 296x - 101$. Find the value of $f(99)$ without using a calculator.

Solution : This is not a calculation you can do directly, even with a calculator (99^6 is a number 12 digits long!). However, the Remainder Theorem says that $f(99)$ is the remainder when $f(x)$ is divided by $(x - 99)$. So instead of evaluating $f(99)$ directly, we divide $f(x)$ by $g(x) = x - 99$ using a long division and determine the remainder this way.

$$
\begin{array}{r|rrrrrrrrrrrrr}
 & x^5 & - & 4x^4 & + & 0x^3 & + & 0x^2 & + & 3x & + & 1 & & \\
\hline
(x-99) & x^6 & - & 103x^5 & + & 396x^4 & + & 0x^3 & + & 3x^2 & - & 296x & - & 101 \\
 & x^6 & - & 99x^5 & & & & & & & & & & \\
 & & - & 4x^5 & & & & & & & & & & \\
 & & - & 4x^5 & + & 396x^4 & & & & & & & & \\
 & & & & & 0 & + & 0 & + & 3x^2 & - & 296x & & \\
 & & & & & & & & & 3x^2 & - & 297x & & \\
 & & & & & & & & & & & x & - & 101 \\
 & & & & & & & & & & & x & - & 99 \\
 & & & & & & & & & & & & & -2 \\
\end{array}
$$

The remainder is -2, so $f(99) = -2$.

Observe that I did not list the lines that lead to $0 \times x^3 + 0 \times x^2$ in the final quotient.

Worked Example 2.19. Suppose we have a polynomial $f(x) = 3x^2 + px - 15$. You are told that $f(x)$ divided by $x + 2$ has remainder of 3. Find p.

Solution :

2.7 Applications of quadratics in science - non examinable

Applications of quadratics in Chemistry

Many times in Chemistry, e.g. when solving equilibrium problems, a quadratic equation results. It has the general form:

$$ax^2 + bx + c = 0$$

Each of the constant terms (a, b, and c) may be positive or negative numbers. A quadratic equation can always be solved by using the quadratic formula (2.2).

There are two roots (answers) to a quadratic equation, because of the ± in the quadratic formula (2.2). In most chemistry problems, only one answer will be meaningful and have physical significance. This means that one answer will make sense, the other answer won't. This will be obvious! Usually when the WRONG answer is plugged in, it will lead to a negative concentration or amount. Since nothing can exist as a negative concentration, the other answer must be the RIGHT one.

Let's work through a typical quadratic calculation that you might find in equilibrium problems.

$$49.0 = \frac{x^2}{(0.300 - x)(0.100 - x)} \tag{2.6}$$

To expand the denominator, multiply the two terms together:

$$(0.300 - x)(0.100 - x) = 0.0300 - 0.400x + x^2$$

Then we have:

$$49.0 = \frac{x^2}{0.0300 - 0.400x + x^2} \tag{2.7}$$

If we cross-multiply, we get:

$$x^2 = 49.0(0.0300 - 0.400x + x^2) = 1.47 - 19.6x + 49.0x^2 \tag{2.8}$$

If we then subtract x^2 from both sides, we can rearrange the equation to get a quadratic equation:

$$0 = 48.0x^2 - 19.6x + 1.47 \tag{2.9}$$

Now, plug the numbers into the quadratic formula, where $a = 48.0, b = -19.6 and c = 1.47$:

$$x = \frac{-(19.6) \pm \sqrt{(-19.6)^2 - 4(48.0)(1.47)}}{2(48.0)} = \frac{19.6 \pm 10.1}{96.0}$$

Therefore, $x = 0.309$ or 0.099.

Chemical Equilibrium Application: At this point, it may be difficult to see which root (answer) is useful and which one is not. Let's look at the original problem:

Consider the following equilibrium having an equilibrium constant = 49.0 at a certain temperature:

$$A(g) + B(g) \leftrightarrows C(g) and D(g)$$

If $0.300 mol$ of A and $0.100 mol$ of B are mixed in a $1.00 litre$ container and allowed to reach equilibrium, what concentrations of A and B will react and what concentrations of C and D will be formed?

In this particular problem, the initial concentrations of two reactants were $0.300M$ and $0.100M$ - these numbers appeared in the denominator of the original problem. The value of x represents the concentration of these reactants that were converted into products. If $0.309M$ of one reactant was lost, that would leave behind $(0.300 - 0.309) = -0.009M$ of one reactant and $(0.100 - 0.309) = -0.209M$ of the other reactant. Since it is impossible to have a negative concentration remaining, the 0.309 number is extraneous (meaningless) and the other, $x = 0.099$ is the root we are interested in. Therefore, A and B both lost $0.099M$ and the equilibrium concentrations of both C and D are $0.099M$.

Worked Example 2.20. Solve for x using the quadratic equation:

$$4.66 \times 10^{-3} = \frac{4x^2}{0.800 - x}$$

Solution :

Final result: $x = 3.00 \times 10^{-2}$ *and* -3.11×10^{-2}. *The obvious root we would be interested in for most chemical applications is the positive number,* 3.00×10^{-2}.

Learning outcomes

- Solving problems related to the number of roots of a quadratic.
- Factorising quadratics.
- Understanding that the roots do not completely determine a quadratic.
- Applying the method of equating coefficients appropriately.
- Finding quadratics whose roots are related to a given polynomial without finding the roots explicitly.
- Applying the Factorisation Theorem to factorise polynomials.
- Applying the method of polynomial division.
- Applying the Remainder Theorem to problems involving polynomials.
- Applying the method of partial fractions to proper and improper polynomial fractions.