Chapter 1

Powers and Logarithms

1.1 Powers or Indices

If $a \in \mathbb{R}$, i.e. a is a real number, and n is a positive integer $1, 2, 3, \ldots$, i.e. $n \in \mathbb{N}$, then $\mathbf{a^n}$ is a short-hand notation for **multiplying a by itself n times**:

$$a^n = \overbrace{a \times \ldots \times a}^{n \text{ times}} . \tag{1.1}$$

The number a is called the **base** and the number n is referred to as the **power**, **index** or **exponent**. The operation of raising a number (base) to a power is called **exponentiation**.

For any **base** a (including zero) and any values of m and n the following hold:

Rules for powers

Rule 1:	$a^1 = a,$		(1.2)
Rule 2:	$a^0 = 1,$		(1.3)
Rule 3:	$a^m \times a^n = a^{m+n},$		(1.4)
Rule 4:	$(a^m)^n = a^{mn},$		(1.5)
Rule 5:	$a^{-1} = \frac{1}{a}$	$(a \neq 0)$	(1.6)
Rule 6:	$a^{-n} = \frac{1}{a^n}$	$(a \neq 0)$	(1.7)
Rule 7:	$a^{1/n} = \sqrt[n]{a}$	$(n \in \mathbb{N}, a > 0)$	(1.8)
Rule 8:	$a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$	$(n \in \mathbb{N}, a > 0)$	(1.9)
Rule 9:	$(ab)^n = a^n b^n$		(1.10)

Remarks:

1) Rules 1, 2, 5 and 7 are in fact definitions so that the rules of powers work nicely.
2) Observe that a is typically given in these expressions, while m and n take different values.

3) Rules 7 and 8 may fail when $a < 0$. $\sqrt{-1}$ is not defined in $\mathbb{R}$, while $\sqrt[3]{-1} = -1$.

Let's take a closer look at Rules 3 and 4 when m and n are positive integer $1, 2, 3, \ldots$:

$$a^m \times a^n = \overbrace{a \times \ldots \times a}^{m \text{ times}} \times \overbrace{a \times \ldots \times a}^{n \text{ times}} = \overbrace{a \times \ldots \times a}^{m+n \text{ times}}$$

and

$$(a^m)^n = \overbrace{\overbrace{a \times \ldots \times a}^{m \text{ times}} \times \ldots \times \overbrace{a \times \ldots \times a}^{m \text{ times}}}^{n \text{ times}} = \overbrace{a \times \ldots \times a}^{m \times n \text{ times}} .$$

While we have demonstrated how Rules 3 and 4 work when m and n take positive integer values we can in fact substitute any fractional or real values into both rules. The same is true for Rule 6, while for Rules 7 and 8 we will only use integer values for m and n.

While these worked examples may appear trivial they are good practice the rules of powers.

Worked Example 1.1. Using the rules for power show that

1. $a \times \frac{1}{a} = 1$
2. $a^n \times \frac{1}{a^n} = 1$
3. $\left(\sqrt[n]{a}\right)^n = a$
4. $\left(\sqrt[n]{a^m}\right)^n = a^m$

Solution :

Common mistakes

You need to remember also that, in general,

$$a^m + a^n \neq a^{m+n} \qquad \text{e.g. } 3^1 + 3^1 = 3 + 3 = 6 \neq 3^{1+1} = 3^2 = 9$$

$$(a+b)^n \neq a^n + b^n \qquad \text{e.g. } (3+4)^2 = 7^2 = 49 \neq 3^2 + 4^2 = 25$$

$$a^{(m^n)} \neq (a^m)^n \qquad \text{e.g. } 2^{(3^2)} = 2^9 = 512 \neq (2^3)^2 = 8^2 = 64.$$

Worked Example 1.2. Without using a calculator find

$$i)\ 2^6, \qquad ii)\ 8^{2/3}, \qquad iii)\ \sqrt[3]{1,000,000}, \qquad iv)\ (0.2)^{-2}$$

Solution :

Worked Example 1.3. Simplify

$$\frac{\sqrt[7]{a^{22}}}{(a^2a^3\sqrt{a})^4}.$$

State the rules of powers that you are using at each step.

Solution : We simplify bit by bit:

Numerator:

Denominator:

Overall fraction:

Worked Example 1.4. Calculate the value of

$$\frac{\sqrt[4]{10^{17}}}{(10^2)^2} \times 10^{\frac{3}{4}}$$

Show your work and state the rules you use.

Solution : Again we simplify bit by bit:

Numerator:

Denominator:

Overall expression:

Observe that expressions may contain **different bases** , e.g. $a^m b^n$. In these cases you **can not apply the rules of powers** until you have transformed all terms to have the same common base using Rule 4. Typically we change to a smaller base.

Example: $9^3 = (3^2)^3 = 3^6$ by Rule 4; and $1000^4 = (10^3)^4 = 10^{12}$ by Rule 4.

Worked Example 1.5. Change 4^7 and 32^2 to the base 2 and hence find $\frac{4^7}{32^2}$.

Solution :

Worked Example 1.6. Show that

$$\left(100^6\right)^{12} \times \left[10^{-8} \times (0.1)^6\right]^{10} = 10000$$

Hint: Write everything in powers of 10.

Solution : Note that a calculator can not help you here (you can try it to see what happens).

First term:

Second term:

Overall expression:

1.2 Exponentials

A common mathematical application for powers are **polynomials** such as $x^7 + 3x^2$. The emphasis lies on the base being a variable, here x, while the power/exponent is fixed, here 7 respectively 2. We will see more on this in later chapters.

However for **exponentials** we view the base as fixed and the exponent as the variable. Fortunately calculations with exponentials follow the rules of powers discussed

Worked Example 1.7. Express $\frac{1}{3^{-4x}}$ in the form a^x, for a suitable number a.

Solution :

Applications of exponentials:

In **biology and ecology** exponentials are used to model unconstrained growth of bacteria. After a fixed unit time, say 1 hour, a cell splits into two new cells. Hence every hour the total number of bacteria doubles. Thus after T hours there are 2^T-times as many bacteria.

In **computer science** Moore's law (See, e.g. on wikipedia.org under "Moore's law") is the observation that the density of transistors in an integrated circuit doubles roughly every two years. Hence in y years the transistor density increases by $2^{(y/2)}$, and in particular it grows by a factor of $\sqrt{2}$ every year.

1.3 Logarithms

In the example on bacterial growth we might ask when are there 8 times as many bacteria as at the start. In other words we are interested in the time T such that $2^T = P$, where $P = 8$.

Worked Example 1.8. i) Find T by recognition such that $2^T = P$, where $P = 8$ or $P = 64$ or $P = 1024$.

Solution :

Remark: The underlying approach of solving exponential problems by *recognition* lies the important idea of *comparing of exponents*.

Suppose in the following Δ and $\square$ are expressions in terms of quantities that we are interested in. Suppose we are asked to solve the equation

$$a^{\Delta} = a^{\square} \tag{1.11}$$

Then the method of *comparing of exponents* states that

$$\Delta = \square \tag{1.12}$$

must be true, i.e. the exponents must agree.

Thus we are able to solve our original equation (1.11) by solving the equation (1.12) instead.

Worked Example 1.9. Find x by recognition such that

a) $7^x = 49$; b) $3^x = 81$; c) $10^x = 100,00$; d) $5^x = 0.2$; e) $2^x = -8$

Solution : a) $49 = 7 \times 7 = 7^2$ so $7^x = 7^2$ gives $x = 2$

Remark: As we have just shown there are problems that we can solve without a calculator just using recognition. We can equally argue that some problems can not have a solution.

The main question that we are trying to answer above is:

Given real numbers a and c, to which power, say x, do we need to raise a to get c?

In other words which number x solves the equation $a^x = c$*?*

Above we solved this problem by recognition, however most times that is not possible for us. In order to invert the operation "raising a to the power of" we apply the operation **logarithm to base** a which is denoted by $\log_a$.

Logarithms to the base a

The equation $a^x = c$ is equivalent to the equation $x = \log_a c$.

Worked Example 1.10. Express the following exponential equations in terms of logarithms

a) $7^x = 49$; b) $3^x = 81$; c) $10^x = 10000$; d) $5^x = 0.2$; e) $2^x = -8$

Solution :

The rules for powers and exponentials can be translated directly into logarithms:

Rules for logarithms

Rule 1:	$\log_a a = 1$		(1.13)
Rule 2:	$\log_a 1 = 0$	$(a \neq 1)$	(1.14)
Rule 3:	$\log_a(xy) = \log_a x + \log_a y$		(1.15)
Rule 4:	$\log_a(x^y) = y \log_a x$		(1.16)
Rule 5:	$\log_a \frac{1}{a} = -1$		(1.17)
Rule 6:	$\log_a \frac{1}{a^n} = -n$		(1.18)
Rule 7:	$\log_a \sqrt[n]{a} = \frac{1}{n}$	$(n \in \mathbb{N})$	(1.19)
Rule 8:	$\log_a \sqrt[n]{a^m} = \frac{m}{n}$	$(n \in \mathbb{N})$	(1.20)

The most common logarithms have the base $a = 10$ and $a = e = 2.71828\ldots$. In order to solve logarithms to another base a we can change the base using the following identity. Suppose we want to express $\log_b$ with base b in terms of $\log_a$ with base a

Rule of change of base for logarithms

$$\log_b c = \frac{\log_a c}{\log_a b}$$

Remarks:

1. Both log terms on the right hand side use base a.
2. On both sides c is in the higher position and b is in the lower position.

In order to remember and understand the rule it helps to look at its derivation:

Worked Example 1.11. [non-examinable but interesting!]
Derive the rule for the change of base.

Worked Example 1.12. Solve $4^x = 64$ using base $a = 4$, $a = 2$ and $a = e$.

Solution :

Worked Example 1.13. Guesstimate the solution of $7^x = 50$ then use a calculator.

Solution :

Worked Example 1.14. Simplify $\log_3 x + \log_9 \left(\frac{1}{x}\right) + \log_{\sqrt{3}} x$ using a base $a = 3$.

Solution :

1.4 Solving equations with exponentials and logarithms

Worked Example 1.15. Given that $\ln a = 2$ solve $a^x = e^6$ for x.

Solution :

Worked Example 1.16. Find x such that $\log_4\left(2^{9x+4}\right) = 4$.

Solution :

1. Use the rules of logarithms

2. Change the base to $a = 2$

We will consider two types of equations with exponentials:

Type 1) Equations containing no sums on either side

Method:

1. Change all terms to a suitable common base, typically the smallest whole number.
2. Combine the terms on either side into a single power.
3. Compare the exponents, respectively the logarithms with respect to the common base.
4. Solve the resulting equation in x.
5. Check your solution solves the original equation.

Worked Example 1.17. Solve $2^{2x} \times 8^{x+1} = 4^{3x}$ for x.
Check your solution does indeed solve the equation.

Solution :

Worked Example 1.18. Show $2^{3x} \times 8^{x+1} = 4^{3x}$ has <u>no</u> solution for x.

Solution :

Type 2) Equations containing sums of exponentials on one side or both

Method:

1. Change all terms to a suitable common base a, typically the smallest whole number.
2. Use the substitution $u = a^x$.
3. Solve the resulting equation, typically a polynomial in u.
4. List the solutions u_i.
5. If $u_i > 0$ then $x_i = \log_a(u_i)$, while $u_i \leq 0$ gives no solution for x.
6. Check your solution solves the original equation.

Worked Example 1.19. Solve $e^{2x} + e^x = 2$ for x. Check your final answer.

Solution :

Worked Example 1.20. Solve $2^{2x} + 8^{x+1} = 4^{2x}$ for x.

Solution :

Worked Example 1.21. Solve the following system of simultaneous equations.

$$\begin{aligned} 2\ln x &= \ln y + \ln 3 \\ e^x e^y &= e. \end{aligned}$$

What restrictions are there on x and y? Explain.

Solution :

1.5 Applications of logarithms in science - non examinable

1) The Carbon-14 dating method

This method measures the percentage of C^{14} isotopes left in bone samples to date them. It uses the fact that the carbon isotope C^{14} has a half life of 5730 years, in other words after 5730 years the amount of C^{14} in a sample has shrunk to 50% of the initial amount.

Let t be the age of the sample measured in years, while τ measures the age in half lives (HL). Hence we have $\tau = 5730\,t$. In particular:

$\tau = 1\text{HL}$, $t = 5730yrs$: 50% of C^{14} left.

$\tau = 2\text{HL}$, $t = 11460yrs$: 25% of C^{14} left.

etc.

The percentage P of C^{14} left after τ HL is given by:

$$P = (0.5)^{\tau}.$$

Hence

$$\tau = \log_{0.5} P = \frac{\log P}{\log 0.5} = \frac{\ln P}{\ln 0.5}$$

Worked Example 1.22. A bone sample is found to have 60% of C^{14} left in it.

- Guesstimate the age of the sample.
- Calculate the exact age of the bone sample.

2) pH-values and the Nernst equation

The pH-value is related to the concentration of the positive hydrogen ions H^+ present in a solution. If $[H^+]$ denotes the concentration of H^+ then the **pH-value** is defined by

$$pH = -\log[H^+].$$

The pH-value is typically measured using electrodes. Let E measure the electrode potential, R be the gas constant, T the temperature in kelvin and F the Faraday constant. Moreover let E_0 measure the standard electrode potential. Then the **Nernst equation** relates the electrode potential E and $[H^+]$ the concentration of hydrogen ions by

$$E = E_0 + \frac{RT}{2F} \ln\left([H^+]^2\right). \tag{1.21}$$

Worked Example 1.23. Rearrange the Nernst equation to get an expression for the pH-value in terms of E, E_0, R, T and F

Learning outcomes

- Knowing and applying the rules for powers and exponentials.
- Simplifying expressions involving powers with the same basis.
- Simplifying expressions involving powers with different bases.
- Understanding the relationship between exponential and logarithmic expressions.
- Transforming exponential expressions into logarithmic ones, and vice versa.
- Knowing and applying the rules for logarithms.
- Knowing and applying the rules for the change of base for logarithms.
- Knowing and applying the rules for logarithms.
- Simplifying logarithmic expressions.
- Understanding and applying the methods for solving different kinds of problems involving exponentials or logarithms.