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1. **(10 points)** `setprobica/multichoice1.pg`
Are the following statements true or false?

? 1. If a linear system has four equations and seven variables, then it must have infinitely many solutions.
? 2. The linear system $A\mathbf{x} = \mathbf{b}$ will have a solution for all $\mathbf{b}$ in $\mathbb{R}^n$ as long as the columns of the matrix A do not include the zero column.
? 3. Every linear system with free variables has infinitely many solutions.
? 4. The linear system $A\mathbf{x} = \mathbf{b}$ will have a solution for all $\mathbf{b}$ in $\mathbb{R}^n$ as long as the columns of the matrix A span $\mathbb{R}^n$
? 5. Different sequences of row operations can lead to different echelon forms for the same matrix.

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2. **(5 points)** `local/Library/TCNJ/TCNJ_RowReduction/problem3.pg`

Determine all values of h and k for which the linear system

$$\begin{array}{rcrcrcr} -3x & - & 3y & - & 3z & = & 4 \\ -8x & - & 9y & - & 8z & = & 7 \\ -35x & - & 39y & + & hz & = & k \end{array}$$

has no solution.

The linear system has no solution if $k \neq$ ___ and $h =$ ___.

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3. **(10 points)** `setprobica/multichoice.pg`
Are the following statements true or false?

? 1. If all the diagonal entries of a square matrix are zero, the matrix is not invertible.
? 2. If A and B are square matrices satisfying $\det(A) = 0$ and $\det(B) = 0$, then $A+B$ cannot be invertible.
? 3. If A is a square matrix satisfying $A^3 = I$, then A is invertible.
? 4. If A is an invertible upper triangular matrix, then A^{-1} is lower triangular.
? 5. If A is a square matrix satisfying $A^2 = O$ (where O is the zero matrix), then $A+I$ is invertible.

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4. **(6 points)** `Library/Rochester/setLinearAlgebra6Determinants/ur_la_6_20.pg`

If

$$\det\begin{bmatrix} a & 1 & d \\ b & 1 & e \\ c & 1 & f \end{bmatrix} = 2, \quad \text{and} \quad \det\begin{bmatrix} a & 1 & d \\ b & 2 & e \\ c & 3 & f \end{bmatrix} = -5,$$

then

$\det\begin{bmatrix} a & 2 & d \\ b & 2 & e \\ c & 2 & f \end{bmatrix} =$ ___ and

$\det\begin{bmatrix} a & 2 & d \\ b & 3 & e \\ c & 4 & f \end{bmatrix} =$ ___.

5. **(5 points)** `local/Library/Hope/Multi1/03-02-Vector-subspaces/Subspaces_nonempty_04.pg`

Let H be the set of all points in the first quadrant in the plane $V = \mathbb{R}^2$. That is, $H = \{(x,y) \mid x \geq 0, y \geq 0\}$.

(1) Is H nonempty?

- choose
- H is empty
- H is nonempty

(2) Is H closed under addition? If it is, enter *CLOSED* . If it is not, enter two vectors in H whose sum is not in H, using a comma separated list and syntax such as `<1,2>, <3,4>`.

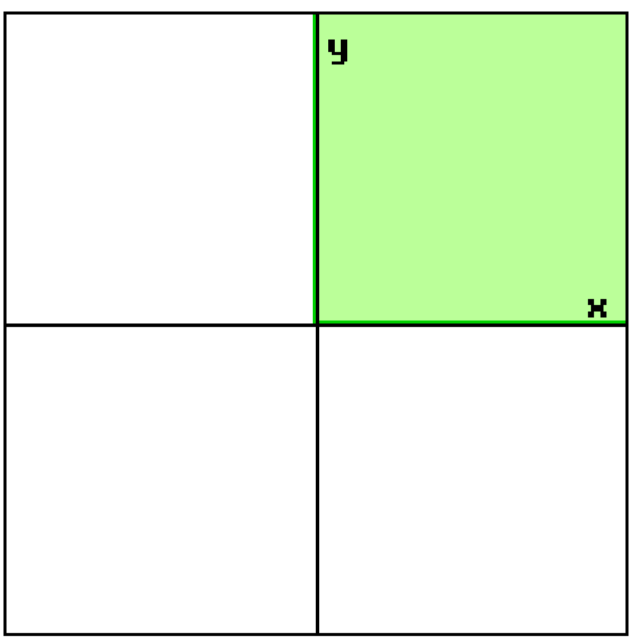
____

(3) Is H closed under scalar multiplication? If it is, enter *CLOSED* . If it is not, enter a scalar in $\mathbb{R}$ and a vector in H whose product is not in H, using a comma separated list and syntax such as `2, <3,4>`.

____

(4) Is H a subspace of the vector space V? You should be able to justify your answer by writing a complete, coherent, and detailed proof based on your answers to parts 1-3.

- choose
- H is a subspace of V
- H is not a subspace of V



6. **(10 points)** `setSemester_A_final_assessment_2020-21/proba.pg`

Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be (non-zero) vectors and suppose that $\mathbf{z} = 2\mathbf{x} - 3\mathbf{y}$ and $\mathbf{w} = 4\mathbf{x} - 6\mathbf{y} - 1\mathbf{z}$.
Are the following statements true or false?

? 1. Span($\mathbf{w}, \mathbf{z}$) = Span($\mathbf{w}, \mathbf{y}$)
? 2. Span($\mathbf{x}, \mathbf{y}$) = Span($\mathbf{w}, \mathbf{z}$)
? 3. Span($\mathbf{x}, \mathbf{y}$) = Span($\mathbf{w}, \mathbf{x}, \mathbf{y}$)
? 4. Span($\mathbf{x}, \mathbf{y}, \mathbf{z}$) = Span($\mathbf{w}, \mathbf{y}, \mathbf{z}$)
? 5. Span($\mathbf{w}, \mathbf{x}, \mathbf{z}$) = Span($\mathbf{w}, \mathbf{x}$)

7. **(10 points)** `Library/TCNJ/TCNJ_LinearIndependence/problem11.pg`

Determine which of the following sets of vectors are linearly independent and which are linearly dependent.

? 1. $\begin{bmatrix} -5 \\ 2 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} -10 \\ 4 \\ -4 \\ 4 \end{bmatrix}$

? 2. $\begin{bmatrix} 6 \\ -10 \\ -6 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \\ 3 \\ -7 \end{bmatrix}, \begin{bmatrix} -5 \\ 10 \\ 7 \\ 2 \end{bmatrix}, \begin{bmatrix} -5 \\ 8 \\ 7 \\ -4 \end{bmatrix}, \begin{bmatrix} 8 \\ -6 \\ -9 \\ 8 \end{bmatrix}$

? 3. $\begin{bmatrix} -7 \\ 3 \\ -6 \\ -10 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ -4 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -8 \\ -9 \\ -5 \end{bmatrix}$

? 4. $\begin{bmatrix} -4 \\ 4 \\ -3 \\ -5 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} -6 \\ 10 \\ -12 \\ -20 \end{bmatrix}$

? 5. $\begin{bmatrix} -1 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -6 \\ 2 \end{bmatrix}, \begin{bmatrix} 8 \\ 7 \\ -14 \\ 4 \end{bmatrix}$

? 6. $\begin{bmatrix} -1 \\ -4 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \\ -9 \\ -16 \end{bmatrix}$

8. **(6 points)** `local/Library/Rochester/setLinearAlgebra14TransfOfRn/ur_la_14_17.pg`

The cross product of two vectors in $\mathbb{R}^3$ is defined by

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}.$$

Let $\mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}$.

Find the matrix A of the linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$ given by $L(\mathbf{x}) = \mathbf{v} \times \mathbf{x}$.

$$A = \begin{bmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{bmatrix}$$

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9. **(10 points)** `setprobica/multichoice2.pg`

Are the following statements true or false for a square matrix A?

? 1. The eigenvalues of a matrix are on its main diagonal.

? 2. An $n \times n$ matrix A is diagonalizable if A has n distinct eigenvectors.

? 3. If $A\mathbf{x} = \lambda\mathbf{x}$ for some vector $\mathbf{x}$ and some scalar λ, then $\mathbf{x}$ is an eigenvector of A.

? 4. Finding an eigenvector of A might be difficult, but checking whether a given vector is in fact an eigenvector is easy.

? 5. If A is invertible, then A is diagonalizable.

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10. **(6 points)** `Library/Rochester/setLinearAlgebra11Eigenvalues/ur_la_11_24a.pg`

The matrix

$$A = \begin{bmatrix} -1 & 1 & 1 & 3 \\ 7 & -1 & 5 & -3 \\ 4 & -1 & 2 & -3 \\ -4 & 1 & 1 & 6 \end{bmatrix}$$

has two distinct real eigenvalues $\lambda_1 < \lambda_2$. Find the eigenvalues and a basis for each eigenspace.

The smaller eigenvalue λ_1 is _____ and a basis for its associated eigenspace is

$$\left\{ \begin{bmatrix} \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \end{bmatrix} \right\}.$$

The larger eigenvalue λ_2 is _____ and a basis for its associated eigenspace is

$$\left\{ \begin{bmatrix} \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \end{bmatrix}, \begin{bmatrix} \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \end{bmatrix} \right\}.$$

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11. **(6 points)** `local/Library/Hope/Multi1/05-04-Diagonalization/DiagR_05.pg`

Suppose

$$A = \begin{bmatrix} -27 & -15 \\ 50 & 28 \end{bmatrix}.$$

Find an invertible matrix P and a diagonal matrix D so that $A = PDP^{-1}$.
Use your answer to find an expression for A^6 in terms of P, a power of D, and P^{-1} in that order.

$A^6 = \begin{bmatrix} \_ & \_ \\ \_ & \_ \end{bmatrix} \begin{bmatrix} \_ & \_ \\ \_ & \_ \end{bmatrix} \begin{bmatrix} \_ & \_ \\ \_ & \_ \end{bmatrix}$.

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12. (10 points) setprobica/multichoice3.pg

Are the following statements true or false?

- [?] 1. If $\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = \|\mathbf{u} - \mathbf{v}\|^2$, then the vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.
- [?] 2. The Gram-Schmidt process produces from a linearly independent set $\{\mathbf{x}_1, ..., \mathbf{x}_p\}$ an orthogonal set $\{\mathbf{v}_1, ..., \mathbf{v}_p\}$ with the property that for each k, the vectors $\mathbf{v}_1, ..., \mathbf{v}_k$ span the same subspace as that spanned by $\mathbf{x}_1, ..., \mathbf{x}_k$.
- [?] 3. For all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, we have $\mathbf{u} \cdot \mathbf{v} = -\mathbf{v} \cdot \mathbf{u}$.
- [?] 4. For any vector $\mathbf{v} \in \mathbb{R}^n$, we have $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|$.
- [?] 5. For a square matrix A, vectors in the column space of A are orthogonal to vectors in the nullspace of A.

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13. (6 points) local/Library/Rochester/setLinearAlgebra18OrthogonalBases/ur_la_18_9.pg

Let

$$A = \begin{bmatrix} -3 & -1 & 0 & 1 \\ 1 & 1 & 2 & -7 \\ 1 & 1 & 2 & -7 \\ 5 & 1 & -2 & 5 \end{bmatrix}.$$

Find **orthonormal** bases of the nullspace of A and the column space of A. Your answers should be correct to 4 decimal places.

Orthonormal basis of the nullspace:

$$\left\{ \begin{bmatrix} \_\_\_\_ \\ \_\_\_ \\ \_\_\_ \\ \_\_\_ \end{bmatrix}, \begin{bmatrix} \_ \\ \_ \\ \_ \\ \_ \end{bmatrix} \right\}.$$

Orthonormal basis of the column space:

$$\left\{ \begin{bmatrix} \_\_\_\_ \\ \_\_\_ \\ \_\_\_ \\ \_\_\_ \end{bmatrix}, \begin{bmatrix} \_ \\ \_ \\ \_ \\ \_ \end{bmatrix} \right\}.$$