Quiz 5: mini problem 1

Question 1.

Suppose $X_1, ..., X_t$ is a sample from a stationary MA(1) time series

$$X_t = 0.2X_{t-1} + \varepsilon_t,$$

where ε_t is an i.i.d. sequence with zero mean and variance 1.

- **(a)** Find the 1-step ahead forecast $\hat{X}_t(1)$ of X_{t+1}, the forecast error and the variance of the forecast error.
- **(b)** Find the 2-step ahead forecast $\hat{X}_t(2)$ of X_{t+2}, the forecast error and the variance of the forecast error.
- **(c)** What can you say about the k-step ahead forecast $\hat{X}_t(1)$ this time series?

Solution. Consider AR(1) time series model

$$X_t = \phi X_{t-1} + \varepsilon_t$$

where $|\phi| < 1$.

(a) We use the following forecasting rules. The k-step ahead forecast is defined by the formula

$$\hat{X}_t(k) = E[X_{t+k}|F_t].$$

We will uses the rules:

$E[\varepsilon_{t+1}|F_t] = 0$, $E[\varepsilon_{t+2}|F_t] = 0$, $E[\varepsilon_{t+3}|F_t] = 0$,...

$E[X_t|F_t] = X_t$, $E[X_{t-1}|F_t] = X_{t-1}$, ... since by time t, we already know X_t, X_{t-1}, ...

$E[\varepsilon_t|F_t] = \varepsilon_t$, $E[\varepsilon_{t-1}|F_t] = \varepsilon_{t-1}$, ...

Recall that we use notation

$$[X_{t+k}] = E[X_{t+k}|F_t]$$

To compute $\hat{X}_t(1)$ first we write

$$X_{t+1} = \phi X_t + \varepsilon_t.$$

Then, using forecasting rules we obtain

$$\hat{X}_t(1) = E[X_{t+1}|F_t] = [X_{t+1}] = [\phi X_t + \varepsilon_{t+1}] = \phi[X_t] + [\varepsilon_{t+1}] = \phi X_t$$

since $[X_t] = E[X_t|F_t] = X_t$ and $[\varepsilon_{t+1}] = E[\varepsilon_{t+1}|F_t] = 0$.

Therefore, the 1-step ahead forecast is $\hat{X}_t(1) = \phi X_t$.

For $\phi = 0.2$, we obtain, $\hat{X}_t(1) = 0.2X_t$.

The error of 1-step ahead forecast is

$$e_t(1) = X_{t+1} - \hat{X}_t(1) = (\phi X_t + \varepsilon_{t+1}) - \phi X_t = \varepsilon_{t+1}.$$

The variance of the error is $Var(e_t(1)) = Var(\varepsilon_{t+1}) = \sigma_\varepsilon^2 = 1$.

(b) To compute 2-step ahead forecast, write

$$X_{t+2} = \phi X_{t+1} + \varepsilon_{t+2}.$$

Then

$$\begin{aligned}\hat{X}_t(2) &= E[X_{t+2}|F_t] = [X_{t+2}] = [\phi X_{t+1} + \varepsilon_{t+2}] = \phi[X_{t+1}] + [\varepsilon_{t+2}] \\ &= \phi X_t(1) + 0 = \phi(\phi X_t) = \phi^2 X_t.\end{aligned}$$

The forecast error:

$$e_t(2) = X_{t+2} - \hat{X}_t(2) = (\phi X_{t+1} + \varepsilon_{t+2}) - \phi X_t(1) = \phi(X_{t+1} - X_t(1)) + \varepsilon_{t+2} = \phi\varepsilon_{t+1} + \varepsilon_{t+2}.$$

The variance of the forecast error

$$Var(e_t(2)) = E(\phi\varepsilon_{t+1} + \varepsilon_{t+2})^2 = E[\phi^2\varepsilon_{t+1}^2] + E[\varepsilon_{t+2}^2] = \phi^2\sigma_\varepsilon^2 + \sigma_\varepsilon^2.$$

We see that

$$Var(e_t(2)) = \phi^2\sigma_\varepsilon^2 + \sigma_\varepsilon^2 \geq Var(e_t(1)) = \sigma_\varepsilon^2$$

that is the variance of the 2-step ahead forecast error is larger than the variance of the 1-step ahead forecast error.

(c) Because of the mean reversion property, as k increases,

$$X_t(k) \to EX_t.$$

The average value of our time series is $EX_t = 0$. Therefore,

$$X_t(k) \to 0.$$