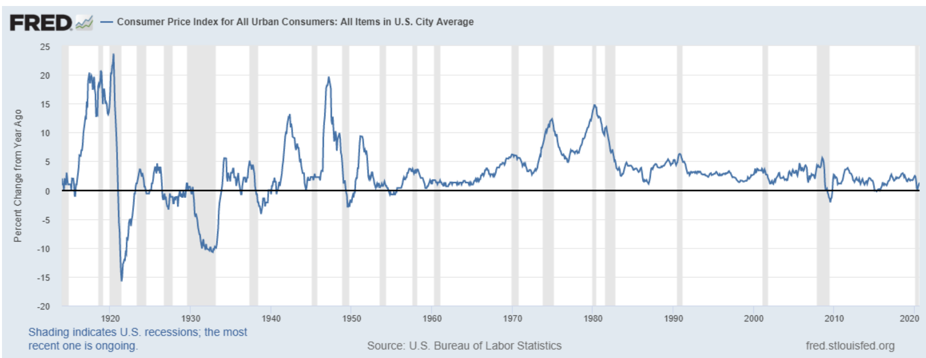
Quiz 6: mini problems 1,2,3

Question 1.

Using given plot, comment whether this time series is stationary or non-stationary.



Solution. This time series does not seem to have a constant mean, so it is a non-stationary time series.

Question 2.

Suppose $Y_1, ..., Y_t$ is a sample from a time series

$$Y_t = Y_{t-1} + x_t, \quad x_t = \phi x_{t-1} + \varepsilon_t$$

where ε_t is an i.i.d. sequence with zero mean and variance 1 and $|\phi| < 1$.

- **(a)** Find the 1-step ahead forecast $\hat{Y}_t(1)$ of Y_{t+1}, the forecast error and the variance of the forecast error.
- **(b)** Find the 2-step ahead forecast $\hat{Y}_t(2)$ of Y_{t+2}, the forecast error and the variance of the forecast error.
- **(c)** Suggest a forecast for Y_{t+20}.

Solution. (a) We have

$$Y_{t+1} = Y_t + X_{t+1}.$$

Note that

$$E[X_{t+1}|F_t] = E[\phi X_t + \varepsilon_{t+1}|F_t] = \phi X_t.$$

Thus

$$\begin{aligned}\hat{Y}_t(1) &= E[Y_{t+1}|F_t] \\ &= E[Y_t + X_{t+1}|F_t] \\ &= E[Y_t|F_t] + E[X_{t+1}|F_t] \\ &= Y_t + \phi X_t.\end{aligned}$$

The 1-step ahead forecast errors is

$$\begin{aligned}e_t(1) &= Y_{t+1} - \hat{Y}_t(1) \\ &= Y_t + X_{t+1} - (Y_t + \phi X_t) \\ &= \varepsilon_{t+1}.\end{aligned}$$

The variance of the 1-step ahead forecast errors is

$$Var(e_t(1)) = Var(\varepsilon_{t+1}) = \sigma_\varepsilon^2.$$

(b) We have

$$Y_{t+2} = Y_{t+1} + x_{t+2}.$$

Note that

$$E[x_{t+2}|F_t] = E[\phi x_{t+1} + \varepsilon_{t+2}|F_t] = E[\phi x_{t+1}|F_t] + E[\varepsilon_{t+2}|F_t] = \phi E[x_{t+1}|F_t] = \phi^2 x_t.$$

Thus

$$\begin{aligned}\hat{Y}_t(2) &= E[Y_{t+2}|F_t] \\ &= E[Y_{t+1} + x_{t+2}|F_t] \\ &= E[Y_{t+1}|F_t] + E[x_{t+2}|F_t] = \hat{Y}_t(1) + \hat{x}_t(2) \\ &= Y_t + \phi x_t + \phi^2 x_t = Y_t + x_t(\phi + \phi^2) = Y_t + (Y_t - Y_{t-1})(\phi + \phi^2)\end{aligned}$$

since $x_t = Y_t - Y_{t-1}$.

The 2-step ahead forecast errors is

$$\begin{aligned} e_t(2) &= Y_{t+2} - \hat{Y}_t(2) \\ &= Y_{t+1} + x_{t+2} - (\hat{Y}_t(1) + \hat{x}_t(2)) \\ &= (Y_{t+1} - \hat{Y}_t(1)) + (x_{t+2} - \hat{x}_t(2)). \end{aligned}$$

We showed that

$$e_t(1) = Y_{t+1} - \hat{Y}_t(1) = \varepsilon_{t+1}.$$

We have

$$\begin{aligned} x_{t+2} - \hat{x}_t(2) &= X_{t+2} - \phi^2 x_t = \phi x_{t+1} + \varepsilon_{t+2} - \phi^2 x_t = \phi(\phi x_t + \varepsilon_{t+1}) + \varepsilon_{t+2} - \phi^2 x_t \\ &= \phi \varepsilon_{t+1} + \varepsilon_{t+2}. \end{aligned}$$

Therefore,

$$e_t(2) = \varepsilon_{t+1} + \phi \varepsilon_{t+1} + \varepsilon_{t+2} = \varepsilon_{t+1}(1+\phi) + \varepsilon_{t+2}.$$

The variance of the 2-step ahead forecast errors is

$$Var(e_t(2)) = Var(\varepsilon_{t+1}(1+\phi) + \varepsilon_{t+2}) = Var(\varepsilon_{t+1}(1+\phi)) + Var(\varepsilon_{t+2}) = (1+\phi)^2\sigma_\varepsilon^2 + \sigma_\varepsilon^2.$$

(c) Since Y_t is a unit root time series, forecasting Y_{t+20}, 20- step ahead would produce a large forecast error. So, not good forecast can be suggested.

Question 3.

Consider time series

$$Y_t = \mu + Y_{t-1} + \varepsilon_{t-1},$$

where ε_t is a white noise sequence with zero mean and variance 1. Suppose that $Y_0 = 1$.

- Find $E[Y_t]$
- $Var(Y_t)$.

Solution.
(a) We can write write

$$\begin{aligned} Y_t &= \mu + Y_{t-1} + \varepsilon_t = \mu + (\mu + Y_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\ &= 2\mu + Y_{t-2} + \varepsilon_{t-1} + \varepsilon_t \\ &= 3\mu + Y_{t-3} + \varepsilon_{t-2} + \varepsilon_{t-1} + \varepsilon_t \\ &= .... \\ &= t\mu + Y_0 + \varepsilon_t + \varepsilon_{t-2} + \varepsilon_{t-1} + \varepsilon_1 \\ &= t\mu + 1 + \varepsilon_t + \varepsilon_{t-2} + \varepsilon_{t-1} + ... + \varepsilon_1, \end{aligned}$$

since $Y_0 = 1$. Then

$$\begin{aligned} E[Y_t] &= E[\mu t + 1 + \varepsilon_t + \varepsilon_{t-2} + ... + \varepsilon_1] \\ &= \mu t + 1 + E[\varepsilon_t] + E[\varepsilon_{t-2}] + ... + E[\varepsilon_1] \\ &= \mu t + 1 + 0 + 0 + ... + 0 = \mu t + 1. \end{aligned}$$

(b)

$$\begin{aligned} Var(Y_t) &= E[(Y_t - E[Y_t])^2] = E[(t\mu + 1 + \varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2} + ... + \varepsilon_1 - \mu t - 1)^2] \\ &= E[(\varepsilon_t + \varepsilon_{t-1} + ... + \varepsilon_1)^2] \\ &= E[\varepsilon_t^2] + E[\varepsilon_{t-1}^2] + ... + E[\varepsilon_1^2] = \sigma_\varepsilon^2 + \sigma_\varepsilon^2 + ... + \sigma_\varepsilon^2 = t\sigma_\varepsilon^2 \end{aligned}$$

noting that ε_t is a white noise and therefore
$E[\varepsilon_i \varepsilon_j] = 0$ if $i \neq j$;
$E[\varepsilon_i \varepsilon_j] = \sigma_\varepsilon^2$ if $i = j$.

(c) By definition,

$$Cov(Y_t, Y_s) = E[(Y_t - E[Y_t])(Y_s - E[Y_s])].$$

Let $t \geq s$.
Since $E[Y_t] = \mu t + 1$ and

$$Y_t - E[Y_t] = \varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2} + ... + \varepsilon_1,$$

we obtain

$$\begin{aligned}
Cov(Y_t, Y_s) &= E[(\varepsilon_t + \varepsilon_{t-1} + ... + \varepsilon_1)(\varepsilon_s + \varepsilon_{s-1} + ... + \varepsilon_1)] \\
&= E[(\{\varepsilon_s + \varepsilon_{t-1} + ... + \varepsilon_1\} + \{\varepsilon_{s+1} + ... + \varepsilon_t\})(\varepsilon_s + \varepsilon_{s-1} + ... + \varepsilon_1)] \\
&= E[(\varepsilon_s + \varepsilon_{t-1} + ... + \varepsilon_1)^2] + E[(\varepsilon_{s+1} + ... + \varepsilon_t)(\varepsilon_s + \varepsilon_{s-1} + ... + \varepsilon_1)] \\
&= s\sigma_\varepsilon^2 + 0 = s\sigma_\varepsilon^2.
\end{aligned}$$

Therefore

$$Cov(Y_t, Y_s) = s\sigma_\varepsilon^2 = \min(t, s)\sigma_\varepsilon^2.$$