CW 11 - SOLUTIONS

① For radial timelike geodesics, we have

$$\mathcal{L} = -\left(1-\frac{2GM}{r}\right)\dot{t}^2 + \frac{\dot{r}^2}{1-\frac{2GM}{r}} = -1$$

Since $\frac{\partial \mathcal{L}}{\partial t} = 0 \Rightarrow \left(1-\frac{2GM}{r}\right)\dot{t} = E$

is conserved. Using this in $\mathcal{L}$ we get

$$\mathcal{L} = -\frac{E^2}{1-\frac{2GM}{r}} + \frac{\dot{r}^2}{1-\frac{2GM}{r}} = -1$$

$$\Rightarrow \quad \dot{r}^2 = \left(1-\frac{2GM}{r}\right)\left[\frac{E^2}{1-\frac{2GM}{r}} - 1\right]$$

$$= E^2 - \left(1-\frac{2GM}{r}\right)$$

Thus, $\frac{dr}{d\tau} = \pm\sqrt{E^2 - \left(1-\frac{2GM}{r}\right)}$

ingoing geodesics

Setting $E = 1$, we have that the ingoing radial geodesics satisfy,

$$\frac{dr}{d\tau} = -\sqrt{\frac{2GM}{r}}$$

Hence the proper time is given by

$$\int_0^\tau d\tau = \tau = -\int_{r_0}^{0} dr \frac{1}{\sqrt{\frac{2GM}{r}}} = \frac{2}{3\sqrt{2GM}} r_0^{3/2}$$

(2) Consider the Kerr metric: $(G=1)$

$$ds^2 = -\frac{\Delta - a^2\sin^2\theta}{\Sigma} dt^2 - \frac{4Mar\sin^2\theta}{\Sigma} dt\,d\phi + \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\Sigma} \sin^2\theta\, d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma\, d\theta^2$$

(Note that $\Sigma - 2Mr = r^2 + a^2\cos^2\theta - 2Mr$

$$= r^2 + a^2 - 2Mr - a^2\sin^2\theta = \Delta - a^2\sin^2\theta$$

The metric can be written as follows:

$$ds^2 = -\frac{\Delta\Sigma}{(a^2+r^2)^2 - a^2\Delta\sin^2\theta} dt^2 + \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\Sigma}\sin^2\theta\left(d\phi - \frac{2Mar}{(r^2+a^2)^2 - a^2\Delta\sin^2\theta} dt\right)^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma\, d\theta^2$$

Following what we did for Schw, we seek a coordinate transformation to cancel the dr^2 term in the metric. Then, we consider

$$d\bar{v} = dt + \frac{f_1(r)}{\Delta} dr$$

Because of the rotation, we also have to transform the angle ϕ if we wish to get rid off the dr^2 term. Then,

$$d\chi = d\phi + \frac{f_2(r)}{\Delta} dr$$

Note that the unknown function f_1 and f_2 can only depend on r, otherwise this wouldn't be a proper coordinate transformation.

Demanding that the crossed term in the metric $d\bar{v}\, d\bar{\phi}$ has coefficient 2 uniquely fixes $f_1(r)$ and $f_2(r)$. We have

$$dt = d\bar{v} - \frac{r^2+a^2}{\Delta} dr$$

$$d\phi = d\chi - \frac{a}{\Delta} dr$$

and the metric becomes:

$$ds^2 = -\frac{\Delta - a^2\sin^2\theta}{\Sigma}\,dv^2 + 2\,dv\,dr$$

$$-2a\sin^2\theta\,\frac{(r^2+a^2-\Delta)}{\Sigma}\,dv\,d\chi - 2a\sin^2\theta\,d\chi\,dr$$

$$+\frac{(r^2+a^2)^2-\Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta\,d\chi^2 + \Sigma\,d\theta^2$$

Note that there is no dr^2 term in the metric, as expected since v is a null coordinate.

You can check that this metric is regular and invertible at $\Delta = 0$.

Consider radial null geodesics on the equatorial plane: $\dot{\chi} = \dot{\theta} = 0$, $\theta = \pi/2$

$$\Rightarrow \mathcal{L} = -\frac{\Delta}{r^2}\dot{v}^2 + 2\dot{v}\dot{r} = 0 \quad \text{for null geods.}$$

$$\dot{v}\left(-\frac{\Delta}{r^2}\dot{v} + 2\dot{r}\right) = 0$$

$\dot{v} = 0 \Rightarrow v = \text{const.} \rightarrow$ ingoing null geods.

$$-\frac{\Delta}{r^2}\dot{v} + 2\dot{r} = 0 \Rightarrow \frac{dv}{dr} = \frac{2r^2}{\Delta} = \frac{2r^2}{(r-r_+)(r-r_-)}$$

$\rightarrow$ outgoing null geods.

We see that at $r = r_+$ (largest real root of Δ) the light cones tilt inwards.

(3)

$$ds^2 = -V\,dt^2 + 2W\,dt\,d\phi + X\,d\phi^2 + e^{2\Lambda}(d\rho^2 + dz^2)$$

The inverse metric is:

$$g^{ab} = \begin{pmatrix} \frac{-X}{VX+W^2} & \frac{W}{VX+W^2} & & 0 \\ \frac{W}{VX+W^2} & \frac{V}{VX+W^2} & & \\ & & e^{-2\Lambda} & \\ 0 & & & e^{-2\Lambda} \end{pmatrix}$$

Therefore,

$$\nabla^a t = g^{ab}\partial_b t = g^{at} = \frac{1}{VX+W^2}\left(-X\,\delta^a{}_t + W\,\delta^a{}_\phi\right)$$

$$g_{ab}\nabla^a t\,\nabla^b t = g_{ab}\,g^{at}g^{bt} = g_{tt}(g^{tt})^2 + 2\,g_{t\phi}\,g^{tt}g^{t\phi} + g_{\phi\phi}(g^{\phi\phi})^2$$
$$= -\frac{X}{VX+W^2}$$

The four-velocity of the locally non-rotating observers is then:

$$u^a = \frac{\nabla^a t}{(-\nabla^b t\,\nabla_b t)^{1/2}} = \frac{1}{\sqrt{X(VX+W^2)}}\left(-X\,\delta^a{}_t + W\,\delta^a{}_\phi\right)$$

The angular momentum of such observers is:

$$L = u^a R_a = g_{ab}\,u^a(\partial_\phi)^b =$$
$$= g_{t\phi}\,u^t + g_{\phi\phi}\,u^\phi =$$
$$= \frac{1}{\sqrt{X(VX+W^2)}}\left(-WX + XW\right) = 0$$

In terms of proper time τ, we have $\dot{x}^a = u^a$, and hence

$$\dot{t} = u^t = -\frac{X}{\sqrt{X(VX+W^2)}}\,, \quad \dot{\phi} = u^\phi = \frac{W}{\sqrt{X(VX+W^2)}}$$

Then, $$\frac{d\phi}{dt} = \frac{\dot{\phi}}{\dot{t}} = -\frac{W}{X}$$

(4) Recall the formula for the surface gravity given in the notes: $\chi^a \nabla_a \chi^b = \kappa \chi^b$

We have

$$\chi_{[a} \nabla_b \chi_{c]} = 0 \quad \text{on the horizon.}$$

Using Killing's equation

$$\nabla_{(a} \chi_{b)} = 0 \Rightarrow \nabla_b \chi_c = - \nabla_c \chi_b$$

$$\Rightarrow \chi_{[a} \nabla_b \chi_{c]} = \chi_a \nabla_b \chi_c + \chi_c \nabla_a \chi_b + \chi_b \nabla_c \chi_a = 0$$

$$\Rightarrow \chi_c \nabla_a \chi_b = -2 \chi_{[a} \nabla_{b]} \chi_c \quad \text{on the horizon.}$$

Contracting with $\nabla^a \chi^b$ we find

$$\chi_c (\nabla^a \chi^b)(\nabla_a \chi_b) = -2 (\chi_a \nabla^a \chi^b)(\nabla_b \chi_c)$$

$$= -2 \kappa \chi^b \nabla_b \chi_c$$

$$= -2 \kappa^2 \chi_c$$

$$\Rightarrow \kappa^2 = -\frac{1}{2} (\nabla^a \chi^b)(\nabla_a \chi_b) \quad \text{on the horizon.}$$

(5) Recall the null coordinates adapted to ingoing and outgoing null geodesics:

$$v = t + r_* \quad \Rightarrow \quad dv = dt + \frac{dr}{f(r)}$$
$$u = t - r_* \quad \Rightarrow \quad du = dt - \frac{dr}{f(r)}$$

where $f(r) = 1 - \frac{2GM}{r}$ for Schwarzschild. The inverse relations are:

$$dt = \frac{1}{2}(dv + du)$$
$$\frac{dr}{f(r)} = \frac{1}{2}(dv - du)$$

Substituting these in the Schwarzschild metric gives

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2_{(2)}$$
$$= -\frac{f(r)}{4}(dv + du)^2 + \frac{f(r)}{4}(dv - du)^2 + r^2 d\Omega^2_{(2)}$$
$$= -f(r)\,dv\,du + r^2 d\Omega^2_{(2)}$$

which coincides with the form of the metric given in the notes.

⑥ In the linearised theory we have

$$g_{ab} = \eta_{ab} + h_{ab} + O(h^2)$$
$$g^{ab} = \eta^{ab} - h^{ab} + O(h^2) \quad \text{where } h^{ab} = \eta^{ac}\eta^{bd} h_{cd}$$

Then,

$$g^{ac} g_{cb} = \left(\eta^{ac} - h^{ac} + O(h^2)\right)\left(\eta_{cb} + h_{cb} + O(h^2)\right)$$
$$= \delta^a{}_b + \cancel{h^a{}_b} - \cancel{h^a{}_b} + O(h^2)$$
$$= \delta^a{}_b + O(h^2)$$

⑦ First we have to compute the Christoffels in the linearised approximation:

$$g_{ab} = \eta_{ab} + h_{ab} + O(h^2)$$
$$g^{ab} = \eta^{ab} - h^{ab} + O(h^2)$$

where indices are raised and lowered with η_{ab}.

Then, the Christoffels are:

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ad}\left(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}\right)$$
$$= \frac{1}{2}\eta^{ad}\left(\partial_b h_{cd} + \partial_c h_{bd} - \partial_d h_{bc}\right) + O(h^2)$$

since $\partial_a \eta_{bc} = 0$

The linearised Riemann tensor is:

$$
\begin{aligned}
R_{abcd} &= g_{ae} R^{e}{}_{bcd} \\
&= g_{ae}\left(\partial_c \Gamma^{e}{}_{bd} - \partial_d \Gamma^{e}{}_{bc} + \Gamma^{e}{}_{fc}\Gamma^{f}{}_{bd} - \Gamma^{e}{}_{fd}\Gamma^{f}{}_{bc}\right) \\
&= \eta_{ae}\left(\partial_c \Gamma^{e}{}_{bd} - \partial_d \Gamma^{e}{}_{bc}\right) \\
&= \frac{1}{2}\partial_c\left(\partial_b h_{da} + \cancel{\partial_d h_{ba}} - \partial_a h_{bd}\right) \\
&\quad - \frac{1}{2}\partial_d\left(\partial_b h_{ca} + \cancel{\partial_c h_{ba}} - \partial_a h_{bc}\right) \\
&= \frac{1}{2}\left(\partial_b\partial_c h_{ad} + \partial_a\partial_d h_{bc} - \partial_a\partial_c h_{bd} - \partial_b\partial_d h_{ca}\right) \\
&\quad + O(h^2)
\end{aligned}
$$

where we have used that $\Gamma\Gamma \sim O(h^2)$

The Ricci tensor at linear order in h_{ab} is:

$$
\begin{aligned}
R_{bd} &= \eta^{ac} R_{abcd} \\
&= \partial^c \partial_{(b} h_{d)c} - \frac{1}{2}\partial^c\partial_c h_{bd} - \frac{1}{2}\partial_b\partial_d h
\end{aligned}
$$

where $h = \eta^{ab} h_{ab}$.

From the Ricci tensor we compute the linearised Ricci scalar:

$$
\begin{aligned}
R &= \eta^{bd} R_{bd} \\
&= \partial^b\partial^c h_{bc} - \partial^2 h
\end{aligned}
$$
where $\partial^2 = \partial^c\partial_c$.

Then, the linearised Einstein tensor is:

$$G_{ab} = R_{ab} - \frac{1}{2}\eta_{ab} R$$

$$= \partial^c \partial_{(a} h_{b)c} - \frac{1}{2}\partial^2 h_{ab} - \frac{1}{2}\partial_a \partial_b h$$

$$- \frac{1}{2}\eta_{ab}\left(\partial^c \partial^d h_{cd} - \partial^2 h\right)$$

which is the desired result.

Now consider the transformation:

$$\bar{h}_{ab} = h_{ab} - \frac{1}{2} h \eta_{ab}$$

$$\bar{h} = \bar{h}^a{}_a = -h$$

$$\Rightarrow \bar{h}_{ab} = h_{ab} + \frac{1}{2}\bar{h}\eta_{ab} \Rightarrow h_{ab} = \bar{h}_{ab} - \frac{1}{2}\bar{h}\eta_{ab}$$

The various terms in G_{ab} in terms of $\bar{h}_{ab}$ become:

$$\partial^c \partial_{(a} h_{b)c} = \partial^c \partial_{(a} \bar{h}_{b)c} - \frac{1}{2}\partial_a \partial_b \bar{h}$$

$$\partial^2 h_{ab} = \partial^2 \bar{h}_{ab} - \frac{1}{2}\eta_{ab}\partial^2 \bar{h}$$

$$\partial_a \partial_b h = -\partial_a \partial_b \bar{h}$$

$$\partial^c \partial^d h_{cd} = \partial^c \partial^d \bar{h}_{cd} - \frac{1}{2}\partial^2 \bar{h}$$

$$\partial^2 h = -\partial^2 \bar{h}$$

Putting everything together, we get

$$G_{ab} = \partial^c \partial_{(a} \bar{h}_{b)c} - \cancel{\frac{1}{2} \partial_a \partial_b \bar{h}}$$

$$- \frac{1}{2} \left(\partial^2 \bar{h}_{ab} - \cancel{\frac{1}{2} \eta_{ab} \partial^2 \bar{h}} \right) + \cancel{\frac{1}{2} \partial_a \partial_b \bar{h}}$$

$$- \frac{1}{2} \eta_{ab} \left[\partial^c \partial^d \bar{h}_{cd} - \cancel{\frac{1}{2} \partial^2 \bar{h}} + \cancel{\partial^2 \bar{h}} \right]$$

$$= - \frac{1}{2} \partial^2 \bar{h}_{ab} + \partial^c \partial_{(a} \bar{h}_{b)c} - \frac{1}{2} \eta_{ab} \partial^c \partial^d \bar{h}_{cd}$$

From which the desired result follows.

(8) Consider the residual gauge transformations:

$$H_{ab} \to H_{ab} + i \left(K_a X_b + K_b X_a - \eta_{ab} K^c X_c \right)$$

The components H_{0a} transform as

$$H_{0a} \to H_{0a} + i \left(K_a X_0 + K_0 X_a \right)$$

Setting this to zero we get:

$$H_{00} + 2i K_0 X_0 = 0 \Rightarrow X_0 = \frac{i H_{00}}{2 K_0}$$

$$H_{0i} + i \left(K_i X_0 + K_0 X_i \right) = 0$$

$$\Rightarrow X_i = \frac{i H_{0i}}{K_0} - \frac{K_i X_0}{K_0} = \frac{i}{K_0} \left(H_{0i} - \frac{K_i H_{00}}{2 K_0} \right)$$

This shows that with this residual gauge freedom we can achieve the longitudinal gauge. However, note that the longitudinal part of the gauge parameter X_a is not fixed. Indeed,

$$K^a X_a = K^0 X_0 + K^i X_i = - K_0 X_0 + K_i X_i$$

$$= - K_0 \left(\frac{i H_{00}}{2K_0} \right) + \frac{i}{K_0} \left(K_i H_{0i} - \frac{K_i K_i}{2K_0} H_{00} \right)$$

$$= - \frac{i H_{00}}{2} + \frac{i}{K_0} \left(K_i H_{0i} - \frac{K_0^2}{2K_0} H_{00} \right)$$

$$= - i H_{00} + \frac{i}{K_0} K_i H_{0i} = \frac{i}{K_0} \left(- K_0 H_{00} + K_i H_{0i} \right)$$

$$= \frac{i}{K_0} \left(+ K^0 H_{00} + K^i H_{0i} \right) = \frac{i}{K_0} K^a H_{0a} = 0$$

since $K^2 = K^a K_a = - K_0^2 + K_i K_i = 0$

The tracelessness condition fixes the longitudinal part of X_a:

$$H = \eta^{ab} H_{ab} \longrightarrow H + i \left(2 K^a X_a - 4 K^c X_c \right)$$

$$= H - 2i K^c X_c = 0$$

$$\Rightarrow \quad K^c X_c = - \frac{i}{2} H$$

(9)

$$\langle \eta^{ab} R^{(2)}_{ab}[h] \rangle = \left| \begin{array}{l}\text{we can ignore the total derivative}\\ \text{in (7.28) in computing the average } \langle \dots \rangle\end{array}\right|$$

$$= \Big\langle \frac{1}{2} h^{cd} \partial^2 h_{cd} - h^{cd} \partial_c \partial^a h_{ad} + \frac{1}{4} (\partial_a h_{cd})(\partial^a h^{cd}) + \frac{1}{2} (\partial^c h^{da})(\partial_c h_{da}) - \frac{1}{2} (\partial^c h^{da})(\partial_d h_{ca}) - \frac{1}{4} (\partial^c h)(\partial_c h) - \left(\partial_c h^{cd} - \frac{1}{2} \partial^d h\right) \partial^a h_{ad} \Big\rangle$$

$$= \Big\langle \cancel{-\frac{1}{2} (\partial_a h_{cd})(\partial^a h^{cd})} + \cancel{(\partial_c h^{cd})(\partial^a h_{ad})} + \frac{1}{4} (\partial_a h_{cd})(\partial^a h^{cd}) + \cancel{\frac{1}{2} (\partial^c h^{da})(\partial_c h_{da})} - \frac{1}{2} (\partial_d h^{da})(\partial^c h_{ca}) - \frac{1}{4} (\partial^c h)(\partial_c h) - \cancel{(\partial_c h^{cd})} (\partial^a h_{ad}) + \frac{1}{2} (\partial^d h)(\partial^a h_{ad}) \Big\rangle$$

$$= \Big\langle \frac{1}{4} (\partial_a h_{cd})(\partial^a h^{cd}) - \frac{1}{2} (\partial_d h^{da})(\partial^c h_{ca}) + \frac{1}{2} (\partial^d h)\left(\partial^a h_{ad} - \frac{1}{2} \partial_d h\right) \Big\rangle$$

$$= \Big\langle -\frac{1}{4} h^{cd} \partial^2 h_{cd} + \frac{1}{2} h^{da} \partial_d \partial^c h_{ca} - \frac{1}{2} h^{ad} \partial_a \partial_d h + \frac{1}{4} h \partial^2 h \Big\rangle$$

$$= \Big\langle \frac{1}{2} h^{cd} \left(-\frac{1}{2} \partial^2 h_{cd} + \partial^a \partial_{(c} h_{d)a} - \frac{1}{2} \partial_c \partial_d h\right) - \frac{1}{4} h^{cd} \partial_c \partial_d h + \frac{1}{4} h \partial^2 h \Big\rangle$$

The term in brackets in the first line is nothing but the linearised Einstein equation in vacuum:

$$R^{(1)}_{ab}[h] = -\frac{1}{2}\partial^2 h_{ab} + \partial^c \partial_{(a} h_{b)c} - \frac{1}{2}\partial_a \partial_b h = 0$$

The second line can be written as the linearised Ricci scalar, which also vanishes in vacuum:

$$R^{(1)}[h] = \eta^{ab} R^{(1)}_{ab}[h] = -\partial^2 h + \partial^a \partial^b h_{ab} = 0$$

$$\Rightarrow \quad \partial^2 h = \partial^a \partial^b h_{ab}.$$

Indeed:

$$\langle \eta^{ab} R^{(2)}_{ab}[h] \rangle = \left\langle -\frac{1}{4} h^{cd} \partial_c \partial_d h + \frac{1}{4} h \partial^2 h \right\rangle$$

$$= \left\langle -\frac{1}{4} h \partial^c \partial^d h_{cd} + \frac{1}{4} h \partial^2 h \right\rangle = 0$$

<u>Note</u>: Throughout this exercise, we have integrated by parts multiple times. We are allowed to do this when computing averages, see the discussion around eq. (7.56)

Now we have to compute $\langle R^{(2)}_{ab}[h] \rangle$ in terms of $\bar{h}_{ab} = h_{ab} - \frac{1}{2}\eta_{ab} h \rightarrow h_{ab} = \bar{h}_{ab} - \frac{1}{2}\eta_{ab}\bar{h}$

Recall that we can ignore the total derivative in the second line in (7.48) when computing the average $\langle\ \rangle$.

The various terms in $R^{(2)}_{ab}[h]$ in terms of $\bar{h}_{ab}$ are:

- $$h^{cd}\partial_a\partial_b h_{cd} = \left(\bar{h}^{cd} - \frac{1}{2}\eta^{cd}\bar{h}\right)\left(\partial_a\partial_b\bar{h}_{cd} - \frac{1}{2}\eta_{cd}\partial_a\partial_b\bar{h}\right)$$
$$= \bar{h}^{cd}\partial_a\partial_b\bar{h}_{cd} - \cancel{\bar{h}\partial_a\partial_b\bar{h}} + \cancel{\bar{h}\partial_a\partial_b\bar{h}} = \bar{h}^{cd}\partial_a\partial_b\bar{h}_{cd}$$
since $\eta^{cd}\eta_{cd} = 4$

- $$h^{cd}\partial_c\partial_a h_{bd} = \left(\bar{h}^{cd} - \frac{1}{2}\bar{h}\eta^{cd}\right)\left(\partial_c\partial_a\bar{h}_{bd} - \frac{1}{2}\eta_{bd}\partial_c\partial_a\bar{h}\right)$$
$$= \bar{h}^{cd}\partial_c\partial_a\bar{h}_{bd} - \frac{1}{2}\bar{h}^c{}_b\partial_a\partial_c\bar{h} - \frac{1}{2}\bar{h}\partial^c\partial_a\bar{h}_{bc} + \frac{1}{4}\bar{h}\partial_b\partial_a\bar{h}$$
$$\Rightarrow h^{cd}\partial_c\partial_{(a}h_{b)d} = \bar{h}^{cd}\partial_c\partial_{(a}\bar{h}_{b)d} - \frac{1}{2}\bar{h}^c{}_{(b}\partial_{a)}\partial_c\bar{h}$$
$$- \frac{1}{2}\bar{h}\partial^c\partial_{(a}\bar{h}_{b)c} + \frac{1}{4}\bar{h}\partial_a\partial_b\bar{h}$$

- $$(\partial_a h_{cd})(\partial_b h^{cd}) = \left(\partial_a\bar{h}_{cd} - \frac{1}{2}\eta_{cd}\partial_a\bar{h}\right)\left(\partial_b\bar{h}^{cd} - \frac{1}{2}\eta^{cd}\partial_b\bar{h}\right)$$
$$= (\partial_a\bar{h}_{cd})(\partial_b\bar{h}^{cd}) - \frac{1}{2}(\partial_a\bar{h})(\partial_b\bar{h}) - \frac{1}{2}(\partial_a\bar{h})(\partial_b\bar{h}) + (\partial_a\bar{h})(\partial_b\bar{h})$$

$$= (\partial_a \bar{h}_{cd})(\partial_b \bar{h}^{cd})$$

- $$(\partial^c h^d{}_b)\partial_c h_{da} = \left(\partial^c \bar{h}^d{}_b - \frac{1}{2}\delta^d{}_b \partial^c \bar{h}\right)\left(\partial_c \bar{h}_{da} - \frac{1}{2}\eta_{da}\partial_c \bar{h}\right)$$

$$= (\partial^c \bar{h}^d{}_b)(\partial_c \bar{h}_{da}) - \frac{1}{2}(\partial^c \bar{h}_{ab})\partial_c \bar{h} - \frac{1}{2}(\partial^c \bar{h})\partial_c \bar{h}_{ab}$$

$$+ \frac{1}{4}\eta_{ab}(\partial^c \bar{h})(\partial_c \bar{h})$$

- $$(\partial^c h^d{}_b)(\partial_d h_{ca}) = \left(\partial^c \bar{h}^d{}_b - \frac{1}{2}\delta^d{}_b \partial^c \bar{h}\right)\left(\partial_d \bar{h}_{ca} - \frac{1}{2}\eta_{ca}\partial_d \bar{h}\right)$$

$$= (\partial^c \bar{h}^d{}_b)(\partial_d \bar{h}_{ca}) - \frac{1}{2}(\partial_a \bar{h}^d{}_b)\partial_d \bar{h} - \frac{1}{2}(\partial^c \bar{h})\partial_b \bar{h}_{ca}$$

$$+ \frac{1}{4}(\partial_a \bar{h})(\partial_b \bar{h}) =$$

$$= (\partial^c \bar{h}^d{}_b)(\partial_d \bar{h}_{ca}) - (\partial^c \bar{h})\partial_{(a}\bar{h}_{b)c} + \frac{1}{4}(\partial_a \bar{h})(\partial_b \bar{h})$$

$$\Rightarrow (\partial^c h^d{}_b)\partial_{[c}h_{d]a}$$

$$= (\partial^c \bar{h}^d{}_b)\partial_{[c}\bar{h}_{d]a} - \frac{1}{2}(\partial^c \bar{h})\left[\partial_c \bar{h}_{ab} - \partial_{(a}\bar{h}_{b)c}\right]$$

$$- \frac{1}{8}\left[(\partial_a \bar{h})(\partial_b \bar{h}) - \eta_{ab}(\partial^c \bar{h})(\partial_c \bar{h})\right]$$

- $$(\partial^c h)(\partial_c h_{ab}) = -(\partial^c \bar{h})\left(\partial_c \bar{h}_{ab} - \frac{1}{2}\eta_{ab}\partial_c \bar{h}\right)$$

$$= -(\partial^c \bar{h})(\partial_c \bar{h}_{ab}) + \frac{1}{2}\eta_{ab}(\partial^c \bar{h})(\partial_c \bar{h})$$

- $$\left(\partial_c h^{cd} - \frac{1}{2}\partial^d h\right)\partial_{(a}h_{b)d} = (\partial_c \bar{h}^{cd})\left(\partial_{(a}\bar{h}_{b)d} - \frac{1}{2}\eta_{d(a}\partial_{b)}\bar{h}\right)$$

$$= (\partial_c \bar{h}^{cd})\partial_{(a}\bar{h}_{b)d} - \frac{1}{2}(\partial^c \bar{h}_{c(a})\partial_{b)}\bar{h}$$

Putting everything together,

$$\langle R^{(2)}_{ab}[h]\rangle =$$

$$= \Big\langle \frac{1}{2}\bar h^{cd}\partial_a\partial_b \bar h_{cd}$$

$$- \bar h^{cd}\cancel{\partial_c}\partial_{(a}\bar h_{b)d} + \frac{1}{2}\bar h^{c}{}_{(b}\cancel{\partial_{a)}}\partial_c \bar h + \frac{1}{2}\bar h\,\partial^c\cancel{\partial_{(a}}\bar h_{b)c} - \frac{1}{4}\bar h\,\partial_a\partial_b \bar h$$

$$+ \frac{1}{4}(\partial_a \bar h_{cd})(\partial_b \bar h^{cd})$$

$$+ (\partial^c \bar h^{d}{}_{b})\,\partial_{[c}\bar h_{d]a} - \frac{1}{2}(\partial^c \bar h)\left[\partial_c \bar h_{ab} - \partial_{(a}\cancel{\bar h}_{b)c}\right]$$

$$- \frac{1}{8}\left[(\partial_a \bar h)(\partial_b \bar h) - \eta_{ab}\cancel{(\partial^c \bar h)(\partial_c \bar h)}\right]$$

$$- \frac{1}{4}\left[-(\partial^c \bar h)(\partial_c \bar h_{ab}) + \frac{1}{2}\cancel{\eta_{ab}(\partial^c \bar h)}(\partial_c \bar h)\right]$$

$$- \left[\cancel{(\partial_c \bar h^{cd})}\,\partial_{(a}\bar h_{b)d} - \frac{1}{2}(\partial^c \bar h_{c\cancel{(a}})\,\partial_{b)}\bar h\right]\Big\rangle$$

$$= \Big\langle -\frac{1}{4}(\partial_a \bar h^{cd})(\partial_b \bar h_{cd}) + \frac{1}{8}(\partial_a \bar h)(\partial_b \bar h)$$

$$- \frac{1}{4}(\partial^c \bar h)(\partial_c \bar h_{ab}) + \frac{1}{2}(\partial^c \bar h^{d}{}_{b})(\partial_c \bar h_{da}) - \frac{1}{2}(\partial^c \bar h^{d}{}_{b})(\partial_d \bar h_{ca})\Big\rangle \qquad (*)$$

Now we use the linearised Einstein eqs. in vacuum:

$$-\frac{1}{2}\partial^c\partial_c \bar h_{ab} + \partial^c\partial_{(a}\bar h_{b)c} - \frac{1}{2}\eta_{ab}\partial^c\partial^d \bar h_{cd} = 0$$

$$-\frac{1}{2}\partial^2 \bar h - \partial^c\partial^d \bar h_{bc} = 0 \Rightarrow \partial^2 \bar h = -2\,\partial^c\partial^d \bar h_{cd}$$

Then the first term in the second line can be written as:

$$\left\langle -\frac{1}{4}(\partial^c \bar{h})(\partial_c \bar{h}_{ab}) \right\rangle = \left\langle +\frac{1}{4}(\partial^2 \bar{h})\bar{h}_{ab} \right\rangle =$$

$$= \left\langle -\frac{1}{2}(\partial^c \partial^d \bar{h}_{cd})\bar{h}_{ab} \right\rangle = \left\langle +\frac{1}{2}(\partial_c \bar{h}^{cd})\partial_d \bar{h}_{ab} \right\rangle$$

The second term in the second line gives:

$$\left\langle \frac{1}{2}(\partial^c \bar{h}_{db})(\partial_c \bar{h}^d{}_a) \right\rangle = \left\langle -\frac{1}{2}(\partial^2 h_{bd})\bar{h}^d{}_a \right\rangle$$

$$= \left\langle \left(-\partial^e \partial_{(b}\bar{h}_{d)e} + \frac{1}{2}\eta_{bd}\,\partial^e \partial^g \bar{h}_{eg}\right)\bar{h}^d{}_a \right\rangle$$

$$= \left\langle (\partial_{(b}\bar{h}_{d)c})\,\partial^c \bar{h}^d{}_a + \frac{1}{2}(\partial^c \partial^d \bar{h}_{cd})\bar{h}_{ab} \right\rangle$$

$$= \left\langle \frac{1}{2}(\partial_b h_{dc})\,\partial^c \bar{h}^d{}_a + \frac{1}{2}(\partial_d h_{bc})(\partial^c \bar{h}^d{}_a) - \right.$$

$$\left. -\frac{1}{2}(\partial^d \bar{h}_{cd})\,\partial^c \bar{h}_{ab} \right\rangle$$

$$= \left\langle \frac{1}{2}(\partial_c h^{cd})(\partial_b h_{ad}) + \frac{1}{2}(\partial_c h_{bd})(\partial^d h^c{}_a) - \frac{1}{2}(\partial^d \bar{h}_{cd})(\partial^c \bar{h}_{ab}) \right\rangle$$

Therefore, the terms in the second line of (*) give:

$$\Big\langle \cancel{+\frac{1}{2}(\partial_c \bar{h}^{cd})\partial_d \bar{h}_{ab}}$$
$$+\frac{1}{2}(\partial_c \bar{h}^{cd})(\partial_{(b}\bar{h}_{a)d}) + \cancel{\frac{1}{2}(\partial_c \bar{h}_{bd})(\partial^d \bar{h}^c{}_a)} \cancel{-\frac{1}{2}(\partial^d \bar{h}_{cd})(\partial^c \bar{h}_{ab})}$$
$$\cancel{-\frac{1}{2}(\partial^c \bar{h}^d{}_b)(\partial_d \bar{h}_{ca})} \Big\rangle$$

$$= \Big\langle \frac{1}{2}(\partial_c \bar{h}^{cd})\,\partial_{(b}\bar{h}_{a)d} \Big\rangle$$

Note that here we have symmetrised the indices (a,b) because $R^{(2)}_{ab}[h]$ is symmetric in (a,b). Hence, we get:

$$\langle R^{(2)}_{ab}[h] \rangle =$$
$$= \Big\langle -\frac{1}{4}(\partial_a \bar{h}^{cd})(\partial_b \bar{h}_{cd}) + \frac{1}{8}(\partial_a \bar{h})(\partial_b \bar{h})$$
$$+\frac{1}{2}(\partial_c \bar{h}^{cd})\,\partial_{(a}\bar{h}_{b)d} \Big\rangle$$

and hence the stress tensor is:

$$\langle t_{ab}[h] \rangle = -\frac{1}{8\pi G}\langle R^{(2)}_{ab}[h] \rangle$$

$$= \frac{1}{32\pi G}\Big\langle (\partial_a \bar{h}^{cd})(\partial_b \bar{h}_{cd}) - \frac{1}{2}(\partial_a \bar{h})(\partial_b \bar{h}) + 2(\partial_c \bar{h}^{cd})\,\partial_{(a}\bar{h}_{b)d} \Big\rangle$$

which is the desired result.

<u>NOTE</u>: this problem (9) is very hard and you should NOT expect that a question similar to this one will be asked in the exam.

This problem is only to illustrate how one does this type of calculations and you can practise playing with the indices.

(10) $$\bar{h}_{ab} = \mathrm{Re}[H_{ab} e^{iK\cdot x}] = H_{ab}\cos(K\cdot x)$$

with $$H_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & H_+ & H_\times & 0 \\ 0 & H_\times & -H_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ and $K^a = \omega(1,0,0,1)$

Notice that $\bar{h}_{ab}$ is already in the transverse and traceless gauge:

$$\partial^c \bar{h}_{ca} = 0 \quad , \quad \bar{h} = 0$$

In this gauge, the expression for $\langle t_{ab}\rangle$ simplifies considerably:

$$\langle t_{ab}\rangle = \frac{1}{32\pi G}\langle (\partial_a \bar{h}_{cd})(\partial_b \bar{h}^{cd})\rangle$$

$$= +\frac{1}{32\pi G} K_a K_b H_{cd} H^{cd} \langle \cos^2(K\cdot x)\rangle$$

Note: $\langle \cos^2(K\cdot x)\rangle = \frac{1}{2}$

$$H_{cd}H^{cd} = 2(H_+^2 + H_\times^2)$$

$K_a = \omega(-1,0,0,1)$, the minus sign coming from lowering the index of K^a.

Therefore, we get :

$$\langle t_{ab} \rangle = \frac{1}{32\pi G} \omega^2 \left(H_+^2 + H_\times^2 \right) \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

(11) Given the setup of the problem, the mass density of the system is :

$$\rho = M \left[\delta(x)\delta(y)\delta(z - z_1(t)) + \delta(x)\delta(y)\delta(z - z_2(t)) \right]$$

with $z_1(t) = \frac{L}{2} + \delta z(t)$, $z_2(t) = -\frac{L}{2} - \delta z(t)$ and $\delta z(t) = A \cos(\omega t)$.

The only non-vanishing component of the quadrupole moment tensor of the energy density is:

$$I_{zz} = \int d^3x \, \rho \, z^2 = 2M \left(\frac{L}{2} + A\cos(\omega t) \right)^2$$

To compute the power, we need the traceless part of the quadrupole moment tensor :

$$Q_{ij} = I_{ij} - \frac{1}{3} \delta_{ij} I_{KK}$$

with $I_{KK} = \sum_{K=1}^{3} I_{KK} = I_{xx} + I_{yy} + I_{zz} = I_{zz}$

$$\Rightarrow \quad Q_{ij} = \begin{pmatrix} -\frac{2M}{3}\left(\frac{L}{2}+A\cos(\omega t)\right)^2 & 0 & 0 \\ 0 & -\frac{2M}{3}\left(\frac{L}{2}+A\cos(\omega t)\right)^2 & 0 \\ 0 & 0 & \frac{4M}{3}\left(\frac{L}{2}+A\cos(\omega t)\right)^2 \end{pmatrix}$$

From this we compute:

$$\dddot{Q}_{ij} = \frac{2}{3} M A \omega^3 \sin(\omega t)\left[L + 8A\cos(\omega t)\right] \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\dddot{Q}_{ij}\dddot{Q}^{ij} = \frac{8}{3} A^2 M^2 \omega^6 \sin^2(\omega t)\left(L + 8A\cos(\omega t)\right)^2$$

$$\langle \dddot{Q}_{ij}\dddot{Q}^{ij} \rangle = \frac{8}{3} A^2 M^2 \omega^6 \Big[L^2 \langle \sin^2(\omega t) \rangle + 16 L A \langle \sin^2(\omega t)\cos(\omega t) \rangle + 16 A^2 \langle \sin^2(2\omega t) \rangle \Big]$$

$$= \frac{8}{3} A^2 M^2 \omega^6 \left[\frac{L^2}{2} + 8A^2\right] = \frac{4}{3} M^2 A^2 L^2 \omega^6 \left(1 + \frac{16A^2}{L^2}\right)$$

where we have used that

$$\langle \sin^2(\omega t) \rangle = \frac{1}{2}$$

$$\langle \sin^2(\omega t)\cos(\omega t) \rangle = \frac{1}{4}\left[\langle \cos(\omega t) \rangle - \langle \cos(3\omega t) \rangle\right] = 0$$

$$\langle \sin^2(2\omega t)\rangle = \frac{1}{2}$$

Using the quadrupole formula we find:

$$\langle P\rangle = \frac{G}{5}\langle \dddot{Q}_{ij}\,\dddot{Q}^{ij}\rangle$$

$$= \frac{4G}{15} M^2 A^2 L^2 \omega^6 \left(1 + \frac{16A^2}{L^2}\right)$$