RELATIVITY – MTH6132

PROBLEM SET 11

Questions **1.-5.** refer to weeks 9-10 material, the rest focuses on weeks 10-12 material. Questions **9** and **10** go beyond what is seen in class and are given just for your own curiosity.

1. Consider the Schwarzschild spacetime. Calculate the proper time that it takes for an infalling observer (i.e., massive particle following a timelike geodesic) in the radial direction to reach $r = 0$ starting from $r = r_0 > 2GM$ with energy $E = 1$.

2. Write Kerr in ingoing EF coordinates. On the equatorial plane, at which value of r do the outgoing radial light cones tilt towards smaller values of r?

3. Consider a stationary, axisymmetric metric of the form

$$ds^2 = -V\,dt^2 + 2\,W\,dt\,d\phi + X\,d\phi^2 + e^{2\Lambda}(d\rho^2 + dz^2)\,,$$

where V, W, X and Λ are functions of ρ and z. We define *locally non-rotating observers* to be the family of observers which are "at rest" with respect to the $t =$ constant hypersurfaces, i.e., whose 4-velocity u^a is proportional to $\nabla^a t = g^{ab}\partial_b t$.

(a) Show that the angular momentum L of such observers vanishes, where L is defined by $L = u^a\,R_a$ and $R^a = (\partial_\phi)^a$ is the rotational Killing vector field.

(b) Show that such observers rotate with a coordinate angular velocity $d\phi/dt = -W/X$. Since, in general, the metric above represents the exterior of a stationary, axisymmetric rotating body (e.g., a black hole), we may interpret this $d\phi/dt$ as resulting from the "dragging of inertial frames" produced by rotating matter.

4. Consider a Killing vector χ^a with a Killing horizon $\mathcal{H}$. Use Killing's equation $\nabla_{(a}\chi_{b)} = 0$ and the fact that $\chi_{[a}\nabla_b\chi_{c]} = 0$ since χ^a is normal to $\mathcal{H}$, to derive the following formula for the surface gravity:

$$\kappa^2 = -\tfrac{1}{2}(\nabla_a\chi_b)(\nabla^a\chi^b)\,.$$

where this formula is to be evaluated at the horizon $\mathcal{H}$.

5. Tranform the Schwarzschild metric to double-null coordinates (u, v).

6. In the linearised theory, show that $g^{ac}g_{cb} = \delta^a{}_b + O(h^2)$.

7. Show that the linearised Einstein tensor is given by

$$G_{ab} = \partial^c\partial_{(a}h_{b)c} - \frac{1}{2}\,\partial^c\partial_c h_{ab} - \frac{1}{2}\,\partial_a\partial_b h - \frac{1}{2}\,\eta_{ab}\left(\partial^c\partial^d h_{cd} - \partial^c\partial_c h\right)\,.$$

Use the transformation $\bar{h}_{ab} = h_{ab} - \frac{1}{2}\, h\, \eta_{ab}$, to show that the linearised Einstein equations are given by

$$-\frac{1}{2}\,\partial^c\partial_c\bar{h}_{ab} + \partial^c\partial_{(a}\bar{h}_{b)c} - \frac{1}{2}\,\eta_{ab}\,\partial^c\partial^d\bar{h}_{cd} = 8\pi\, G\, T_{ab}\,.$$

8. Show that the residual gauge freedom can be used to achieve a "longitudinal" gauge,

$$H_{0a} = 0\,.$$

Further, show that the remaining gauge freedom can be used to achieve the additional "trace-free" condition,

$$H^a{}_a = 0\,.$$

9. Using the linearised Einstein equation show that, in vacuum,

$$\langle \eta^{ab} R^{(2)}_{ab}[h]\rangle = 0\,,$$

and hence the second term in $t_{\mu\nu}[h]$ averages to zero. Furthermore, show that

$$\langle t_{ab}\rangle = \frac{1}{32\pi\, G}\langle(\partial_a\bar{h}_{cd})\partial_b\bar{h}^{cd} - \frac{1}{2}(\partial_a\bar{h})\partial_b\bar{h} - 2\,(\partial_c\bar{h}^{cd})\partial_{(a}\bar{h}_{b)d}\rangle\,.$$

Note: *This problem is very hard and it is NOT the type of problem that you should expect in a quiz or in the exam. I suggest that you look at the solution to see what type of computations are done in this context and practice manipulating the indices.*

10. Calculate $\langle t_{ab}\rangle$ for the plane gravitational wave that we have been considering, namely eq. (7.17) in the notes. In particular, show that

$$\langle t_{ab}\rangle = \frac{1}{32\pi\, G}\left(H_+^2 + H_\times^2\right) k_a k_b \langle\cos^2(k_c x^c)\rangle = \frac{\omega^2}{32\pi\, G}\left(H_+^2 + H_\times^2\right)\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore, as expected, there is a flux of energy and momentum travelling at the speed of light in the z-direction.

11. Consider two equal masses M at the ends of a massless spring of length L. The masses oscillate along the z-direction with amplitude A so that the centre of mass is fixed and at the origin of a Cartesian coordinate system. In this setup, the masses only move along the z-direction and their trajectories are $z_1(t) = \frac{L}{2} + \delta z(t)$ and $z_2(t) = -\frac{L}{2} - \delta z(t)$ respectively, where $\delta z(t) = A\,\cos(\omega\, t)$. Calculate, using the quadrupole formula, the power radiated in gravitational waves by this system.