SOLUTIONS: CW3

(1) Only the second and the fourth make sense. In the first one, the free indices don't match. In the third expression the index k appears as a dummy index on the left while it's a free index on the right. In the fifth, the free indices don't match and k appears as both a free and a dummy index on the r.h.s.

(2) G_{ij}: $G_{11}, G_{12}, G_{21}, G_{22}$

$A^i B_i = A^1 B_1 + A^2 B_2$

$\Gamma^i{}_{jk}$: $\Gamma^1{}_{11}, \Gamma^1{}_{12}, \Gamma^1{}_{21}, \Gamma^1{}_{22}, \Gamma^2{}_{11}, \Gamma^2{}_{12}, \Gamma^2{}_{21}, \Gamma^2{}_{22}$

$\Gamma^i{}_{ij}$: $\Gamma^1{}_{11} + \Gamma^2{}_{21}, \Gamma^1{}_{12} + \Gamma^2{}_{22}$

$R^i{}_{jkl}$: $R^1{}_{111}, R^1{}_{112}, R^1{}_{121}, R^1{}_{122}$

$R^1{}_{211}, R^1{}_{212}, R^1{}_{221}, R^1{}_{222}$

$R^2{}_{111}, R^2{}_{112}, R^2{}_{121}, R^2{}_{122}$

$R^2{}_{211}, R^2{}_{212}, R^2{}_{221}, R^2{}_{222}$

$R^i{}_i = R^1{}_1 + R^2{}_2$

③ The first part is covered in the lectures.

- Timelike: $\bar{A}\cdot\bar{A} < 0$
- Spacelike: $\bar{A}\cdot\bar{A} > 0$
- Null: $\bar{A}\cdot\bar{A} = 0$

For the second part,

$\bar{A}$: timelike $\Rightarrow$ $\bar{A}\cdot\bar{A} = -(A^0)^2 + |\underline{A}|^2 < 0$

$\Rightarrow$ $|\underline{A}| < A^0$ and $A^0 > 0$ wlog

Here $|\underline{A}| = \sqrt{(A^1)^2 + (A^2)^2 + (A^3)^2}$ is the usual norm of a 3-vector.

$\bar{B}$: null $\Rightarrow$ $\vec{B}\cdot\bar{B} = -(B^0)^2 + |\underline{B}|^2 = 0 \Rightarrow B^0 = \pm|\underline{B}|$

We can choose the + sign wlog.

Then,

$$\underline{A}\cdot\underline{B} = -A^0B^0 + \underline{A}\cdot\underline{B} < -|\underline{A}||\underline{B}| + \underline{A}\cdot\underline{B} =$$

$$= -|\underline{A}||\underline{B}| + |\underline{A}||\underline{B}|\cos\theta =$$

$$= -|\underline{A}||\underline{B}|(1-\cos\theta)$$

$$\Rightarrow \underline{A}\cdot\underline{B} < -|\underline{A}||\underline{B}|(1-\cos\theta) < 0$$

(4) If $\bar{A}$ is a unit spacelike vector, then $|\bar{A}| = 1$.

It follows then $-(A^0)^2 + 4 = 1 \Rightarrow A^0 = \pm\sqrt{3}$.

If $\bar{A}$ and $\bar{B}$ are orthogonal, then

$$\bar{A}\cdot\bar{B} = -3A^0 + 2B^2 = 0$$

$$\Rightarrow B^2 = \frac{3}{2}A^0 = \pm\frac{3\sqrt{3}}{2}$$

(5) a) $\bar{A}\cdot\bar{B} = \eta_{ab}A^a B^b$

$|\bar{A}|^2$ is invariant means that is the same in all reference frames, i.e., it is unchanged by a Lorentz transf.

b) Shown in the lectures.

c) $|\bar{A}|^2, |\bar{B}|^2 > 0$, $\bar{A}\cdot\vec{B} = 0$

$$(\bar{A}+\bar{B})^2 = |\bar{A}|^2 + |\bar{B}|^2 + 2\cancelto{0}{\bar{A}\cdot\bar{B}}$$

$$= |\bar{A}|^2 + |\bar{B}|^2 > 0$$

(6) Since $\bar{A}$ and $\bar{B}$ are timelike,

$$|\bar{A}|^2 = -(A^0)^2 + (A^1)^2 < 0, \quad |\bar{B}|^2 = -(B^0)^2 + (B^1)^2 < 0$$

Hence, $A^1 < A^0$ and $B^1 < B^0$ since A^0, A^1, B^0, B^1 are all positive. Adding,

$$(A^1 + B^1)^2 < (A^0 + B^0)^2$$

Hence,

$$|\bar{A} + \bar{B}|^2 = -(A^0 + B^0)^2 + (A^1 + B^1)^2 < 0$$

(7) Since $\bar{A}$ is a 4-vector, its transformation under Lorentz transf. is given by

$$A'^0 = \gamma(A^0 - v A^1), \quad A'^1 = \gamma(A^1 - v A^0), \quad A'^2 = A^2, \quad A'^3 = A^3$$

The norm of $\bar{A}'$ is given by

$$|\bar{A}'|^2 = -(A'^0)^2 + (A'^1)^2 + (A'^2)^2 + (A'^3)^2 =$$

$$= -\gamma^2(A^0 - vA^1)^2 + \gamma^2(A^1 - vA^0)^2 + (A^2)^2 + (A^3)^2$$

$$= \gamma^2(1 - v^2)(-(A^0)^2 + (A^1)^2) + (A^2)^2 + (A^3)^2$$

$$= -(A^0)^2 + (A^1)^2 + (A^2)^2 + (A^3)^2$$

since $\gamma^2(1 - v^2) = 1$

(8) The relation between Cartesian and polar coordinates is

$$x = r\cos\theta \quad , \quad y = r\sin\theta \quad \Rightarrow \quad r = \sqrt{x^2+y^2} \quad , \quad \theta = \arctan\frac{y}{x}$$

Taking differentials,

$$dx = \cos\theta\, dr - r\sin\theta\, d\theta$$

$$dy = \sin\theta\, dr + r\cos\theta\, d\theta$$

Therefore,

$$ds^2 = dx^2 + dy^2 =$$

$$= (\cos\theta\, dr - r\sin\theta\, d\theta)^2 + (\sin\theta\, dr + r\cos\theta\, d\theta)^2$$

$$= dr^2 + r^2\, d\theta^2$$

$$F_{tr} = -F_{rt} = -\frac{Q}{r^2}$$

$$T_{ab} = F_{ac} F_b{}^c - \frac{1}{4} g_{ab} F_{cd} F^{cd}$$

$$\eta = -dt^2 + dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$$

$$\rightarrow \quad F^{tr} = -F^{rt} = \eta^{ta}\eta^{rb} F_{ab} = \eta^{tt}\eta^{rr} F_{tr} = (-1) F_{tr} = \frac{Q}{r^2}$$

$$F_{ab} F^{ab} = F_{tr} F^{tr} + F_{rt} F^{rt} = -\frac{2Q^2}{r^4}$$

The first term only has tt and rr components. Indeed,

$$F_{tc} F_r{}^c = F_{tr} F_r{}^r = F_{tr} F_{rb} \eta^{br} = F_{tr} F_{rr} \eta^{rr} = 0$$

$$F_{ta} F_t{}^a = F_{tr} F_t{}^r = F_{tr} F_{tr} \eta^{rr} = +\frac{Q^2}{r^4}$$

$$F_{ra} F_r{}^a = F_{rt} F_r{}^t = F_{rt} F_{rt} \eta^{tt} = -\frac{Q^2}{r^4}$$

Therefore we get:

$$T_{tt} = F_{ta} F_t{}^a - \frac{1}{4} \eta_{tt} F^2 = \frac{Q^2}{r^4} - \frac{1}{4}(-1)\left(-\frac{2Q^2}{r^4}\right) = \frac{Q^2}{2r^4}$$

$$T_{rr} = F_{ra} F_r{}^a - \frac{1}{4} \eta_{rr} F^2 = -\frac{Q^2}{r^4} - \frac{1}{4}\left(-\frac{2Q^2}{r^4}\right) = -\frac{Q^2}{2r^4}$$

$$T_{\theta\theta} = -\frac{1}{4} \eta_{\theta\theta} F^2 = -\frac{1}{4}(r^2)\left(-\frac{2Q^2}{r^4}\right) = +\frac{Q^2}{2r^2}$$

$$T_{\phi\phi} = \sin^2\theta \, T_{\theta\theta}$$

$$T = T^a{}_a = \eta^{ab} T_{ab} = \eta^{tt} T_{tt} + \eta^{rr} T_{rr} + \eta^{\theta\theta} T_{\theta\theta} + \eta^{\phi\phi} T_{\phi\phi} =$$
$$= (-1)\left(\frac{Q^2}{2r^4}\right) + \left(-\frac{Q^2}{2r^4}\right) + \frac{1}{r^2}\left(\frac{Q^2}{2r^2}\right) + \frac{1}{r^2 \sin^2\theta}\left(\frac{Q^2}{2r^2}\sin^2\theta\right) = 0$$

① Question: Geometry

- Consider the object $F_{ij} = \partial_i A_j - \partial_j A_i$, where A_i is a $(0,1)$ tensor. Compute the transformation of F_{ij} under a change of coordinates. $x^a = x^a(x'^b)$. Is F_{ij} a tensor?

- Solution:

Since A_i is a tensor, it transforms as

$$A_i = \frac{\partial x'^j}{\partial x^i} A'_j$$

As for the partial derivatives one has,

$$\frac{\partial}{\partial x^i} = \frac{\partial x'^j}{\partial x^i} \frac{\partial}{\partial x'^j}$$

Putting these two results together, we get

$$\begin{aligned}
F_{ij} &= \partial_i A_j - \partial_j A_i \\
&= \frac{\partial x'^k}{\partial x^i} \frac{\partial}{\partial x'^k} \left(\frac{\partial x'^l}{\partial x^j} A'_l \right) - \frac{\partial x'^p}{\partial x^j} \frac{\partial}{\partial x'^p} \left(\frac{\partial x'^q}{\partial x^i} A'_q \right) \\
&= \frac{\partial x'^k}{\partial x^i} \left[\frac{\partial^2 x'^l}{\partial x'^k \partial x^j} A'_l + \frac{\partial x'^l}{\partial x^j} \frac{\partial}{\partial x'^k} A'_l \right]
\end{aligned}$$

$$- \frac{\partial x'^p}{\partial x^j}\left[\frac{\partial^2 x'^q}{\partial x'^p \partial x^i} A'_q + \frac{\partial x'^q}{\partial x^i}\frac{\partial}{\partial x'^p} A'_q \right]$$

The first terms on each line cancel. To see this, one realises that they are equal to

$$\frac{\partial x'^k}{\partial x^i}\frac{\partial}{\partial x'^k}\left(\frac{\partial x'^l}{\partial x^j}\right) A'_l = \frac{\partial^2 x'^l}{\partial x^i \partial x^j} A'_l$$

$$\frac{\partial x'^p}{\partial x^j}\frac{\partial}{\partial x'^p}\left(\frac{\partial x'^q}{\partial x^i}\right) A'_q = \frac{\partial^2 x'^q}{\partial x^j \partial x^i} A'_q$$

Therefore, one is left with,

$$\begin{aligned} F_{ij} &= \frac{\partial x'^k}{\partial x^i}\frac{\partial x'^l}{\partial x^j}\frac{\partial A'_l}{\partial x'^k} - \frac{\partial x'^p}{\partial x^j}\frac{\partial x'^q}{\partial x^i}\frac{\partial}{\partial x'^p} A'_q \\ &= \frac{\partial x'^k}{\partial x^i}\frac{\partial x'^l}{\partial x^j}\left(\frac{\partial A'_l}{\partial x'^k} - \frac{\partial A'_k}{\partial x'^l}\right) \\ &= \frac{\partial x'^k}{\partial x^i}\frac{\partial x'^l}{\partial x^j} F'_{lk} \end{aligned}$$

Therefore, F_{ij} transforms as a $(0,2)$ tensor.