③ Under Galilean transf.

$$t' = t \quad , \quad x' = x - vt \quad \text{or} \quad t = t' \ , \ x = x' + vt$$

Hence,

$$\frac{\partial}{\partial t} = \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} + \frac{\partial x'}{\partial t}\frac{\partial}{\partial x'} = \frac{\partial}{\partial t'} - v\frac{\partial}{\partial x'}$$

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial t}\right) = \left(\frac{\partial}{\partial t'} - v\frac{\partial}{\partial x'}\right)\left(\frac{\partial \phi}{\partial t'} - v\frac{\partial \phi}{\partial x'}\right) =$$

$$= \frac{\partial^2 \phi}{\partial t'^2} - 2v\frac{\partial^2 \phi}{\partial t' \partial x'} + v^2 \frac{\partial^2 \phi}{\partial x'^2}$$

since $\dfrac{\partial^2 \phi}{\partial t' \partial x'} = \dfrac{\partial^2 \phi'}{\partial x' \partial t'}$

Similarly,

$$\frac{\partial}{\partial x} = \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'} + \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} = \frac{\partial}{\partial x'} \Rightarrow \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi}{\partial x'^2}$$

Hence,

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x'^2} - \frac{1}{c^2}\left(\frac{\partial^2 \phi}{\partial t'^2} - 2v\frac{\partial^2 \phi}{\partial t' \partial x'} + v^2\frac{\partial^2 \phi}{\partial x'^2}\right)$$

$$\neq \frac{\partial^2 \phi}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t'^2}$$

$\Rightarrow$ The wave equation is not invariant under Galilean transf.

Under Lorentz transf. we have:

$$\frac{\partial}{\partial t} = \gamma\left(\frac{\partial}{\partial t'} - v\frac{\partial}{\partial x'}\right) \quad , \quad \frac{\partial}{\partial x} = \gamma\left(\frac{\partial}{\partial x'} - \frac{v}{c^2}\frac{\partial}{\partial t'}\right)$$

Hence,

$$\frac{\partial^2\phi}{\partial t^2} = \gamma^2\left(\frac{\partial}{\partial t'} - v\frac{\partial}{\partial x'}\right)\left(\frac{\partial\phi}{\partial t'} - v\frac{\partial\phi}{\partial x'}\right) =$$

$$= \gamma^2\left(\frac{\partial^2\phi}{\partial t'^2} - 2v\frac{\partial^2\phi}{\partial t'\partial x'} + v^2\frac{\partial^2\phi}{\partial x'^2}\right)$$

$$\frac{\partial^2\phi}{\partial x^2} = \gamma^2\left(\frac{\partial}{\partial x'} - \frac{v}{c^2}\frac{\partial}{\partial t'}\right)\left(\frac{\partial\phi}{\partial x'} - \frac{v}{c^2}\frac{\partial\phi}{\partial t'}\right) =$$

$$= \gamma^2\left(\frac{\partial^2\phi}{\partial x'^2} - \frac{2v}{c^2}\frac{\partial^2\phi}{\partial t'\partial x'} + \frac{v^2}{c^4}\frac{\partial^2\phi}{\partial t'^2}\right)$$

$$\Rightarrow \frac{\partial^2\phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \gamma^2\left(1 - \frac{v^2}{c^2}\right)\left(\frac{\partial^2\phi}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t'^2}\right) =$$

$$= \frac{\partial^2\phi}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t'^2}$$

since $\gamma^2\left(1 - \frac{v^2}{c^2}\right) = 1$

Hence, the wave equation is indeed invariant under Lorentz transf.

(4) Under a vectorial Galilean transf.

$$\underline{r}' = \underline{r} - \underline{v}t \,, \quad t' = t$$

we have

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - \underline{v}\cdot\nabla' \,, \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial x'} \,, \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial y'} \,, \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial z'}$$

Thus,

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \nabla'^2\phi - \frac{1}{c^2}\left(\frac{\partial}{\partial t'} - \underline{v}\cdot\nabla'\right)\left(\frac{\partial\phi}{\partial t'} - \underline{v}\cdot\nabla'\phi\right)$$

$$= \nabla'^2\phi - \frac{1}{c^2}\left(\frac{\partial^2\phi}{\partial t'^2} - 2\,\underline{v}\cdot\nabla'\left(\frac{\partial\phi}{\partial t'}\right) + \underline{v}\cdot\nabla'\left(\underline{v}\cdot\nabla'\phi\right)\right)$$

$\Rightarrow$ the wave equation is not invariant.

(5) Starting from the Lorentz transf,

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \quad , \quad x' = \gamma(x - vt)$$

Adding and subtracting gives,

$$ct' - x' = \varepsilon\,(ct - x) \quad , \quad ct' + x' = \varepsilon\,(ct + x)$$

where $\varepsilon = \sqrt{\frac{1 + v/c}{1 - v/c}}$

To show that the combination of two LT's is another LT, consider a frame F'' moving with respect to F' with velocity v'. Then,

$$ct'' - x'' = \varepsilon'\,(ct' - x') \quad , \quad ct'' + x'' = \varepsilon'\,(ct' + x')$$

with $\varepsilon' = \sqrt{\frac{1 + v'/c}{1 - v'/c}}$. Hence,

$$ct'' - x'' = \varepsilon' \cdot \varepsilon\,(ct - x) \quad , \quad ct'' + x'' = \varepsilon' \cdot \varepsilon\,(ct' + x')$$

$$\varepsilon' \cdot \varepsilon = \sqrt{\frac{1 + v'/c}{1 - v'/c}}\sqrt{\frac{1 + v/c}{1 - v/c}} =$$

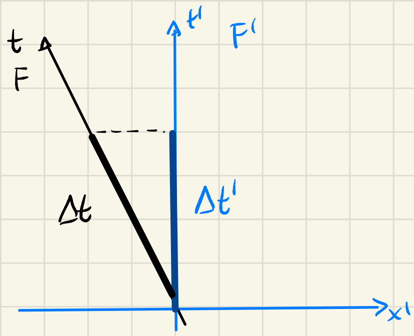
$$= \sqrt{\frac{1 + v''/c}{1 - v''/c}} \qquad \text{where } v'' = \frac{v + v'}{1 + \frac{vv'}{c^2}}$$

$$\Rightarrow \varepsilon' \cdot \varepsilon = \varepsilon''$$

(6) Covered in the lectures:

By the inverse Lorentz transf:

$$\Delta t = \gamma\left(\Delta t' + \frac{v\Delta x'}{c^2}\right) = \gamma \Delta t' \quad \text{since} \quad \Delta x' = 0$$

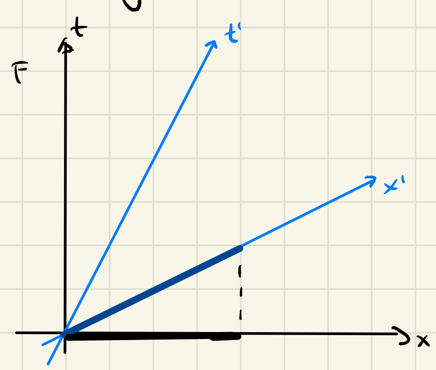


(7) Covered in the lectures.

By the Lorentz transf,

$$\Delta x' = \gamma(\Delta x - v\Delta t) = \gamma \Delta x \Rightarrow \Delta x = \frac{1}{\gamma}\Delta x'$$

since $\Delta t = 0$ because the distance between the ends of the rod is measured simultaneously in F.



(8) The half-life of the pions is $\tau = 1.8 \cdot 10^{-8}$ s in their rest frame. This means that if at $t=0$ there N pions, after time τ there will be only N/2 pions left.

For an observer in the lab, the pions' clock runs slow because of time dilation. The half-life of moving pions in the lab is

$$\Delta t = \gamma \Delta t' = \frac{1.8 \cdot 10^{-8}\, s}{\sqrt{1-(0.996)^2}} = 20.1 \cdot 10^{-8}\, s.$$

Thus, in the lab pions will travel

$$\Delta x = v \Delta t = 0.996 \cdot (3 \cdot 10^{8}\, m/s) \cdot (20.1 \cdot 10^{-8}\, s) = 60.2\, m$$

before half of them decay.

Without time dilation, they would travel

$$\Delta x = 0.996 \cdot (3 \cdot 10^{8}\, m/s) \cdot (1.8 \cdot 10^{-8}\, s) = 5.4\, m$$

before half of them decay.

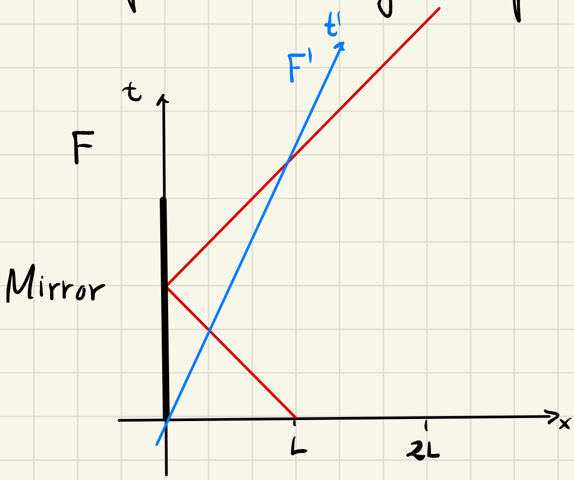
(9) The train has length $\Delta x' = L$ in its own reference frame F'. From the lectures, we have seen that the length contraction is given by

$$\Delta x = \frac{1}{\gamma} \Delta x'$$

So, for $\Delta x = L/3$ we have

$$\frac{L}{3} = \frac{1}{\gamma} L = \sqrt{1 - \frac{v^2}{c^2}}\, L \Rightarrow \boxed{\frac{v}{c} = \frac{2\sqrt{2}}{3}}$$

⑩ Spacetime diagram from the point of view of Joe:



Spacetime diagram from Moe's point of view: