Queen Mary University of London

Main Examination period 2023 – May/June – Semester B

MTH5113: Introduction to Differential Geometry

Duration: 2 hours

(Solutions)

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

The exam is intended to be completed within **2 hours**. However, you will have a period of **4 hours** to complete the exam and submit your solutions.

You should attempt ALL questions. Marks available are shown next to the questions.

You are allowed to bring **three A4 sheets of paper** as notes for the exam.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiners: A. Shao, E. Katirtzoglou

Question 1 [23 marks]. Consider the curve

$$C = \{(x, y) \in \mathbb{R}^2 \mid (x-1)^2 + 4(y+1)^2 = 4\},$$

and consider the following parametrisation of C:

$$\gamma : \mathbb{R} \to C, \qquad \gamma(t) = (1 + 2\cos t,\ -1 + \sin t).$$

(a) Sketch the image of γ. **[6]**

(b) Find the unit normals to C at the point $(1, -2)$. **Draw and label these on your sketch from part (a).** **[8]**

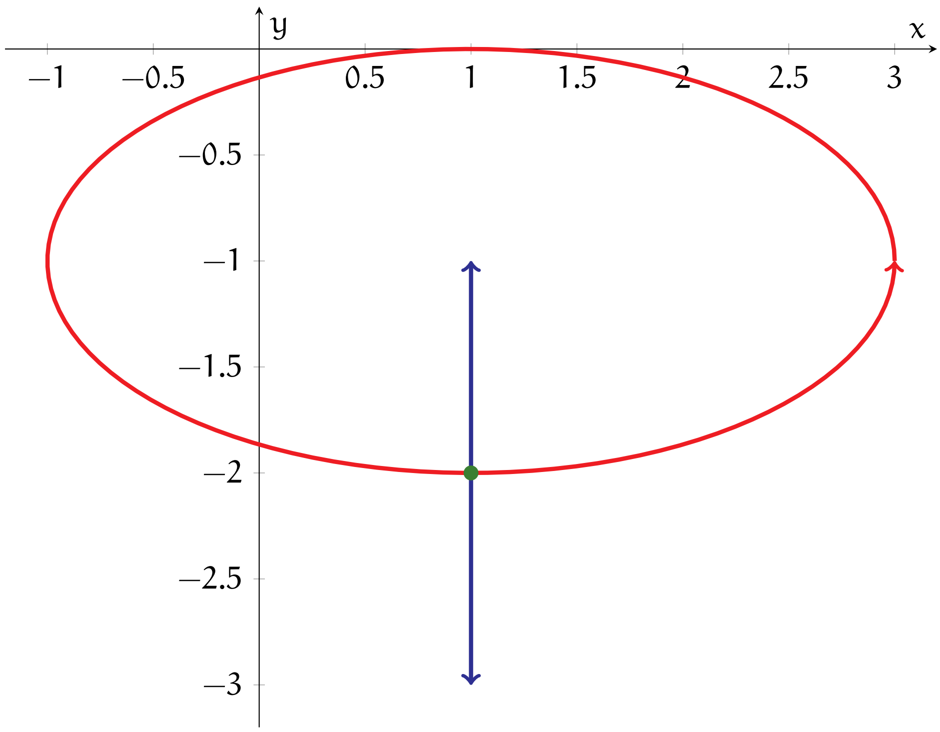
(c) Assume C is also given the clockwise orientation. Compute the curve integral

$$\int_C \mathbf{F} \cdot d\mathbf{s},$$

where $\mathbf{F}$ is the vector field on $\mathbb{R}^2$ given by

$$\mathbf{F}(x, y) = (-y, x)_{(x,y)}.$$ **[9]**

(a) [Seen similar] The image of γ is drawn in red. [6 marks] (Here, one needs not be exact—the general shape of γ along with a few key values of γ would suffice.)



(b) [Seen] First, note that $(1,-2)$ corresponds to

$$(1,-2)=\gamma\left(-\tfrac{\pi}{2}\right). \quad [1 \text{ mark}]$$

Taking a derivative, we see that

$$\gamma'(t)=(-2\sin t,\ \cos t), \qquad \gamma'\left(-\tfrac{\pi}{2}\right)=(2,\ 0), \qquad \left|\gamma'\left(-\tfrac{\pi}{2}\right)\right|=2. \quad [2 \text{ marks}]$$

Thus, the unit tangents are given by

$$\mathbf{t}^{\pm}=\pm\frac{\gamma'\left(-\frac{\pi}{2}\right)}{\left|\gamma'\left(-\frac{\pi}{2}\right)\right|}=\pm(1,\ 0)_{(1,-2)}. \quad [2 \text{ marks}]$$

By rotating, we obtain the unit normals

$$\mathbf{n}^{\pm}=\pm(0,\ 1)_{(1,-2)}. \quad [1 \text{ mark}]$$

The answer can also be obtained via the gradient of $(x-1)^2+4(y+1)^2$. [3 marks for correct gradient, 3 marks for applying level set theorem correctly.]

The unit normals are drawn as blue arrows on the sketch in part (a). [2 marks]

(c) [Seen similar] Consider the restricted parametrisation of C given by

$$\lambda:(0,2\pi)\to C, \qquad \lambda(t)=(1+2\cos t,-1+\sin t).$$

Note that λ is injective, and its image is all of C except for a single point. [2 marks] Moreover, note that λ generates the anticlockwise orientation of C, which is opposite of our given orientation. As a result, we have that

$$\int_C \mathbf{F}\cdot d\mathbf{s}=-\int_\lambda \mathbf{F}\cdot d\mathbf{s}. \quad [2 \text{ marks}]$$

Next, we compute the necessary quantities:

$$\begin{aligned}
\mathbf{F}(\lambda(t)) &= (1-\sin t,\ 1+2\cos t)_{(1+2\sin t,-1+\cos t)},\\
\lambda'(t)_{\lambda(t)} &= (-2\sin t,\ \cos t)_{(1+2\sin t,-1+\cos t)},\\
\mathbf{F}(\lambda(t))\cdot\lambda'(t)_{\lambda(t)} &= -2\sin t+2\sin^2 t+\cos t+2\cos^2 t\\
&= 2-2\sin t+\cos t. \quad [3 \text{ marks}]
\end{aligned}$$

Thus, putting all the above together, we compute that

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{s} &= -\int_0^{2\pi}(2-2\sin t+\cos t)\,dt\\
&= -\left.(2t+2\cos t+\sin t)\right|_{t=0}^{t=2\pi}\\
&= -4\pi. \quad [2 \text{ marks}]
\end{aligned}$$

Question 2 [25 marks]. Consider the surface

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid (x+z)^2 + (y+z)^2 = 1,\ 0 < z < 2\},$$

and consider the following parametrisation of S:

$$\sigma : \mathbb{R} \times (0, 2) \to S, \qquad \sigma(u, v) = (-v + \cos u,\ -v + \sin u,\ v).$$

(a) Sketch the image of σ. Moreover, on your sketch, indicate (i) one path obtained by holding v constant and varying u, and (ii) one path obtained by holding u constant and varying v. **[8]**

(b) Find the tangent plane to S at the point $(-1, 0, 1)$. **[7]**

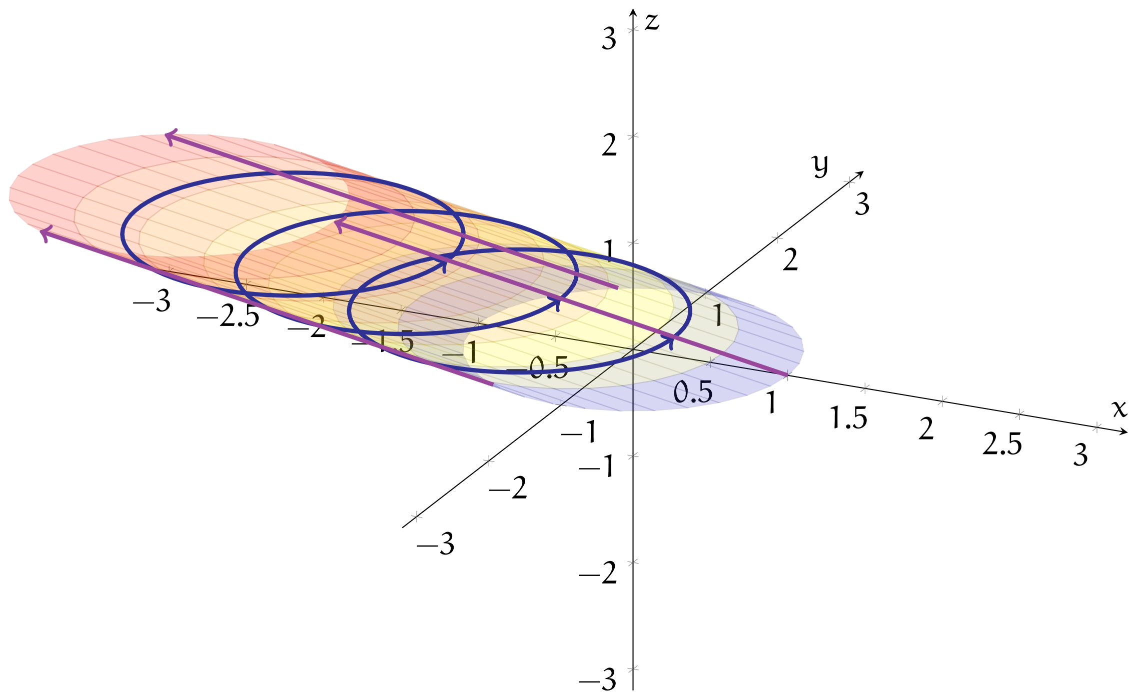
(c) Compute the surface integral

$$\iint_S F\, dA,$$

where F is the real-valued function given by

$$F(x, y, z) = \frac{1}{\sqrt{1 + (x+z)(y+z)}}, \quad \text{where } 1 + (x+z)(y+z) > 0.$$ **[10]**

(a) [Seen similar] The image of σ is drawn below [4 marks]; examples of level paths are drawn in (i) blue [2 marks] and (ii) purple [2 points], respectively. (Here, one needs not be exact—getting the general shape of σ along with a few key values of σ would suffice.)



(b) [Seen] The partial derivatives of σ satisfy

$$\partial_1\sigma(u,v) = (-\sin u,\, \cos u,\, 0), \qquad \partial_2\sigma(u,v) = (-1,-1,1). \quad \text{[2 marks]}$$

Noting that $(-1,0,1) = \sigma\left(1,\frac{\pi}{2}\right)$ [2 marks], we then compute

$$\partial_1\sigma\left(1,\,\frac{\pi}{2}\right) = (-1,0,0), \qquad \partial_2\sigma\left(1,\,\frac{\pi}{2}\right) = (-1,-1,1). \quad \text{[2 marks]}$$

As a result, the tangent plane is given by

$$\begin{aligned} T_{(-1,0,1)}S &= T_\sigma\left(1,\,\frac{\pi}{2}\right) \\ &= \left\{a \cdot (-1,0,0)_{(-1,0,1)} + b \cdot (-1,-1,1)_{(-1,0,1)} \middle| a,b \in \mathbb{R}\right\}. \quad \text{[1 mark]} \end{aligned}$$

(c) [Seen similar] To obtain an appropriate parametrisation, we restrict σ to

$$\rho : (0,2\pi) \times (0,2) \to S, \qquad \rho(u,v) = (-v + \cos u,\, -v + \sin u,\, v).$$

Note that ρ is injective, and its image is all of S except for a line. [2 marks]

Moreover, direct computations (see also (b)) yield

$$\partial_1\rho(u,v) = (-\sin u,\, \cos u,\, 0), \qquad \partial_2\rho(u,v) = (-1,-1,1), \quad \text{[1 mark]}$$

as well as

$$\begin{aligned} \partial_1\rho(u,v) \times \partial_2\rho(u,v) &= (\cos u,\, \sin u,\, \cos u + \sin u), \\ |\partial_1\rho(u,v) \times \partial_2\rho(u,v)| &= \sqrt{\cos^2 + \sin^2 u + (\cos u + \sin u)^2} \\ &= \sqrt{2 + 2\cos u \sin u}, \\ F(\rho(u,v)) &= \frac{1}{\sqrt{1 + \cos u \sin u}} \quad \text{[4 marks]} \end{aligned}$$

Combining the above, we then obtain

$$\begin{aligned} \iint_S F\, dA &= \iint_{(0,2\pi)\times(0,2)} \sqrt{2 + 2\cos u \sin u} \frac{1}{\sqrt{1 + \cos u \sin u}}\, du dv \\ &= \sqrt{2} \int_0^{2\pi} du \int_0^2 dv \\ &= 4\sqrt{2}\pi. \quad \text{[3 marks]} \end{aligned}$$

Question 3 [20 marks]. Using the method of Lagrange multipliers, find the minimum and maximum values of the function

$$f : \mathbb{R}^3 \to \mathbb{R}, \qquad f(x, y, z) = x + z,$$

subject to the constraint

$$x^2 + 4y^2 + z^2 = 4.$$

Also, at which points are these minimum and maximum values achieved? [20]

[Seen similar] First, notice the constraint surface $x^2 + 4y^2 + z^2 = 4$ (an ellipsoid) is both closed and bounded, thus the maximum and minimum values in this constrained optimisation problem are guaranteed to exist. [2 marks]

Let g denote the function corresponding to the constraint:

$$g : \mathbb{R}^3 \to \mathbb{R}, \qquad g(x, y, z) = x^2 + 4y^2 + z^2.$$

First, we compute the gradients of f and g:

$$\nabla f(x, y, z) = (1, 0, 1)_{(x,y,z)}, \qquad \nabla g(x, y, z) = (2x,\ 8y,\ 2z)_{(x,y,z)}.$$

The method of Lagrange multipliers then indicates that we must solve the system,

$$1 = \lambda \cdot 2x, \qquad 0 = \lambda \cdot 8y, \qquad 1 = \lambda \cdot 2z, \qquad x^2 + 4y^2 + z^2 = 4.$$ [4 marks]

First, note from the first and third equations that none of x, z, λ can be zero, or we obtain a contradiction $1 = 0$ [2 marks]. To solve the system, we now split into cases:

- If $y \neq 0$, then $8\lambda y \neq 0$, and the second equation yields a contradition. Thus, the case $y \neq 0$ yields no solutions to our system. [2 marks]
- If $y = 0$, then the second equation trivially holds, and the fourth equation yields

  $$x^2 + z^2 = 4.$$

  Moreover, the first and third equations imply

  $$\frac{1}{2x} = \lambda = \frac{1}{2z}, \qquad x = z.$$

  Combining all the above equations then yields

  $$2x^2 = 4, \qquad z = x = \pm\sqrt{2}.$$

  Recalling that $\lambda = \frac{1}{2x}$, we see that the above yields two solutions to the system:

  $$(x, y, z, \lambda) = \left(+\sqrt{2}, 0, +\sqrt{2}, +\frac{1}{2\sqrt{2}}\right), \left(-\sqrt{2}, 0, -\sqrt{2}, -\frac{1}{2\sqrt{2}}\right).$$ [6 marks]

Thus, our system has precisely two solutions, as listed above.

We can now evaluate f at each of the above solution points:

$$f(+\sqrt{2}, 0, +\sqrt{2}) = +2\sqrt{2}, \qquad f(-\sqrt{2}, 0, -\sqrt{2}) = -2\sqrt{2}.$$ [2 marks]

Since the extrema of f are guaranteed to exist, we can hence conclude that:

- The maximum is $+2\sqrt{2}$, achieved at $(x, y, z) = (+\sqrt{2}, 0, +\sqrt{2})$. [1 mark]
- The minimum is $-2\sqrt{2}$, achieved at $(x, y, z) = (-\sqrt{2}, 0, -\sqrt{2})$. [1 mark]

Question 4 [18 marks].

(a) Give a parametrisation of the curve
$$P = \{(x, y) \in \mathbb{R}^2 \mid x^2 - 2xy = -4\}$$
whose image contains the point $(2, 2)$. [**6**]

(b) Is the following set connected:
$$E = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq |x| \leq 2\}?$$
Briefly justify your answer. (You may draw E to aid in this.) [**6**]

(c) Consider the unit circle
$$\mathcal{C} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$
For each of the following parametrisations of $\mathcal{C}$, state whether it generates the anticlockwise or clockwise orientation of $\mathcal{C}$:

(i) $\gamma_1 : \mathbb{R} \to \mathcal{C}$, where $\gamma_1(t) = (-\cos t, \sin t)$.

(ii) $\gamma_2 : (-1, 1) \to \mathcal{C}$, where $\gamma_2(t) = \left(t, \sqrt{1 - t^2}\right)$.

(iii) $\gamma_3 : (-1, 1) \to \mathcal{C}$, where $\gamma_3(t) = \left(\sqrt{1 - t^2}, t\right)$. [**6**]

(a) [Seen] One way to do this is to set $t = x$, and to note that
$$y = \frac{t^2 + 4}{2t} = \frac{t}{2} + \frac{2}{t}.$$
This results in the following parametrisation:
$$\gamma : (0, \infty) \to P, \qquad \gamma(t) = \left(t, \frac{t}{2} + \frac{2}{t}\right).$$
(Note that $(2, 2) = \gamma(2)$.) [4 marks for correct formula] [2 marks for correct domain]

(b) [Unseen] Note that E consists of two disconnected strips,
$$E_- = \{(x, y) \in \mathbb{R}^2 \mid -2 \leq x \leq -1\}, \qquad E_+ = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x \leq 2\} \quad \text{[3 marks]}.$$
[Alternatively, 3 marks for a correct sketch of E.]

Observe that one cannot go from a point $\mathbf{p} \in E_-$ to another point $\mathbf{q} \in E_+$ without leaving E. (In particular, the x-coordinate must be between -1 and 1 at some point.) As a result, E is not connected. [3 marks]

(c) [Seen similar]

- γ_1 generates the clockwise orientation of $\mathcal{C}$ [2 marks]
- γ_2 generates the clockwise orientation of $\mathcal{C}$ [2 marks]
- γ_3 generates the anticlockwise orientation of $\mathcal{C}$ [2 marks]

Question 5 [14 marks]. Consider the vector fields $\mathbf{F}$ and $\mathbf{G}$ on $\mathbb{R}^3$ given by

$$\mathbf{F}(x, y, z) = (-y + xz,\ x + yz,\ 0)_{(x,y,z)}, \qquad \mathbf{G}(x, y, z) = (-y,\ x,\ 2)_{(x,y,z)}.$$

(a) Show that $\nabla \times \mathbf{F} = \mathbf{G}$. [5]

(b) Let $\mathcal{E}$ be the half-sphere

$$\mathcal{E} = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1,\ z > 0\},$$

and suppose $\mathcal{E}$ is given the outward-facing (i.e. upward-facing) orientation. Apply **Stokes' theorem** and part (a) to show that

$$\iint_{\mathcal{E}} \mathbf{G} \cdot d\mathbf{A} = 2\pi. \qquad [9]$$

(a) [Seen] This is a direct computation using the definition of curl:

$$\begin{aligned}
&(\nabla \times \mathbf{F})(x, y, z) \\
&\quad = (\partial_y 0 - \partial_z(x + yz),\ \partial_z(-y + xz) - \partial_x 0,\ \partial_x(x + yz) - \partial_y(-y + xz))_{(x,y,z)} \quad \text{[3 marks]} \\
&\quad = (-y,\ x,\ 2)_{(x,y,z)}. \quad \text{[2 marks]}
\end{aligned}$$

(b) [Unseen] Let $\mathcal{C}$ denote the boundary of $\mathcal{E}$, the equatorial circle:

$$\mathcal{C} = \{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}.$$

Then, by Stokes' theorem and part (a), we obtain that

$$\iint_{\mathcal{E}} \mathbf{G} \cdot d\mathbf{A} = \iint_{\mathcal{E}} (\nabla \times \mathbf{F}) \cdot d\mathbf{A} = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}, \quad \text{[4 marks]}$$

where $\mathcal{C}$ is given the positive orientation relative to $\mathcal{E}$, which here is the anticlockwise orientation when viewed from above.

To evaluate the above curve integral, we note that

$$\gamma : (0, 2\pi) \to \mathcal{C}, \qquad \gamma(t) = (\cos t,\ \sin t,\ 0)$$

is an injective parametrisation of $\mathcal{C}$ whose image is all of $\mathcal{C}$ except for a single point, and that γ generates the above-mentioned positive orientation of $\mathcal{C}$. [2 marks] As a result,

$$\begin{aligned}
\iint_{\mathcal{E}} \mathbf{G} \cdot d\mathbf{A} &= +\int_0^{2\pi} [\mathbf{F}(\gamma(t)) \cdot \gamma'(t)]\, dt \\
&= \int_0^{2\pi} [(-\sin t + \cos t \cdot 0,\ \cos t + \sin t \cdot 0,\ 0) \cdot (-\sin t,\ \cos t,\ 0)]\, dt \quad \text{[2 marks]} \\
&= \int_0^{2\pi} dt \\
&= 2\pi. \quad \text{[1 mark]}
\end{aligned}$$

End of Paper.