Queen Mary
University of London

Main Examination period 2023 – May/June – Semester B

MTH5113: Introduction to Differential Geometry

Duration: 2 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

The exam is intended to be completed within **2 hours**. However, you will have a period of **4 hours** to complete the exam and submit your solutions.

You should attempt ALL questions. Marks available are shown next to the questions.

You are allowed to bring **three A4 sheets of paper** as notes for the exam.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

Possession of unauthorised material at any time when under examination conditions is an assessment offence and can lead to expulsion from QMUL. Check now to ensure you do not have any unauthorised notes, mobile phones, smartwatches or unauthorised electronic devices on your person. If you do, raise your hand and give them to an invigilator immediately.

It is also an offence to have any writing of any kind on your person, including on your body. If you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms, it will be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. A mobile phone that causes a disruption in the exam is also an assessment offence.

Exam papers must not be removed from the examination room.

Examiners: A. Shao, E. Katirtzoglou

Question 1 [23 marks]. Consider the curve

$$C = \{(x, y) \in \mathbb{R}^2 \mid (x-1)^2 + 4(y+1)^2 = 4\},$$

and consider the following parametrisation of C:

$$\gamma : \mathbb{R} \to C, \qquad \gamma(t) = (1 + 2\cos t,\ -1 + \sin t).$$

(a) Sketch the image of γ. [**6**]

(b) Find the unit normals to C at the point $(1, -2)$. **Draw and label these on your sketch from part (a)**. [**8**]

(c) Assume C is also given the clockwise orientation. Compute the curve integral

$$\int_C \mathbf{F} \cdot d\mathbf{s},$$

where $\mathbf{F}$ is the vector field on $\mathbb{R}^2$ given by

$$\mathbf{F}(x, y) = (-y, x)_{(x,y)}.$$ [**9**]

Question 2 [25 marks]. Consider the surface

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid (x+z)^2 + (y+z)^2 = 1,\ 0 < z < 2\},$$

and consider the following parametrisation of S:

$$\sigma : \mathbb{R} \times (0, 2) \to S, \qquad \sigma(u, v) = (-v + \cos u,\ -v + \sin u,\ v).$$

(a) Sketch the image of σ. Moreover, on your sketch, indicate (i) one path obtained by holding v constant and varying u, and (ii) one path obtained by holding u constant and varying v. [**8**]

(b) Find the tangent plane to S at the point $(-1, 0, 1)$. [**7**]

(c) Compute the surface integral

$$\iint_S F\, dA,$$

where F is the real-valued function given by

$$F(x, y, z) = \frac{1}{\sqrt{1 + (x+z)(y+z)}}, \quad \text{where } 1 + (x+z)(y+z) > 0.$$ [**10**]

Question 3 [20 marks]. Using the method of Lagrange multipliers, find the minimum and maximum values of the function

$$f : \mathbb{R}^3 \to \mathbb{R}, \qquad f(x, y, z) = x + z,$$

subject to the constraint

$$x^2 + 4y^2 + z^2 = 4.$$

Also, at which points are these minimum and maximum values achieved? [**20**]

Question 4 [18 marks].

(a) Give a parametrisation of the curve

$$P = \{(x, y) \in \mathbb{R}^2 \mid x^2 - 2xy = -4\}$$

whose image contains the point $(2, 2)$. [**6**]

(b) Is the following set connected:

$$E = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq |x| \leq 2\}?$$

Briefly justify your answer. (You may draw E to aid in this.) [**6**]

(c) Consider the unit circle

$$\mathcal{C} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

For each of the following parametrisations of $\mathcal{C}$, state whether it generates the anticlockwise or clockwise orientation of $\mathcal{C}$:

(i) $\gamma_1 : \mathbb{R} \to \mathcal{C}$, where $\gamma_1(t) = (-\cos t, \sin t)$.

(ii) $\gamma_2 : (-1, 1) \to \mathcal{C}$, where $\gamma_2(t) = \left(t, \sqrt{1 - t^2}\right)$.

(iii) $\gamma_3 : (-1, 1) \to \mathcal{C}$, where $\gamma_3(t) = \left(\sqrt{1 - t^2}, t\right)$. [**6**]

Question 5 [14 marks]. Consider the vector fields $\mathbf{F}$ and $\mathbf{G}$ on $\mathbb{R}^3$ given by

$$\mathbf{F}(x, y, z) = (-y + xz,\ x + yz,\ 0)_{(x,y,z)}, \qquad \mathbf{G}(x, y, z) = (-y,\ x,\ 2)_{(x,y,z)}.$$

(a) Show that $\nabla \times \mathbf{F} = \mathbf{G}$. **[5]**

(b) Let $\mathcal{E}$ be the half-sphere

$$\mathcal{E} = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1,\ z > 0\},$$

and suppose $\mathcal{E}$ is given the outward-facing (i.e. upward-facing) orientation. Apply **Stokes' theorem** and part (a) to show that

$$\iint_{\mathcal{E}} \mathbf{G} \cdot d\mathbf{A} = 2\pi.$$ **[9]**

End of Paper.