Queen Mary University of London

MTH5114 **Linear Programming and Games, Spring 2024**

Week 11 Seminar Questions **Viresh Patel**

Practice Exam Question:

Consider the following 2-player zero-sum game. Each player separately chooses a number from the set $\{1, 2, 3\}$. Both players then reveal their numbers. If the numbers match, the row player must pay £3 to the column player, otherwise, the player with the lower number must pay £1 to the player with the higher number.

(a) Give the payoff matrix for this game from the perspective of the row player. Also give the security level for each of the player's strategies.

Solution:

	1	2	3
1	-3	-1	-1
2	1	-3	-1
3	1	1	-3

The security levels for the row player's strategies (1, 2, and 3) are -3, -3 and -3, respectively. The security levels for the column player's strategies are 1, 1, and -1, respectively.

(b) Does this game possess a pure Nash equilibrium? If so, give all pure Nash equilibria for the game. If not, say why.

Solution: The game does not possess a pure Nash equilibrium, because the maximum security level for the row player is -3 but the minimum security level for the column player is -1. At a Nash equilibrium, these two quantities must be equal.

(c) Formulate a linear program that finds the row player's best mixed strategy in this game (you do not need to solve this program). [You will be able to do this part of the question at the end of week 11.]

Solution:

$$\begin{aligned}
\text{maximize} \quad & s \\
\text{subject to} \quad & s \leq -3x_1 + x_2 + x_3, \\
& s \leq -x_1 - 3x_2 + x_3, \\
& s \leq -x_1 - x_2 - 3x_3, \\
& x_1 + x_2 + x_3 = 1, \\
& x_1, x_2, x_3 \geq 0, \\
& s \text{ unrestricted}
\end{aligned}$$

Discussion Questions:

1. Consider the 2-player, zero-sum game "Rock, Paper, Scissors". Each player chooses one of 3 strategies: *rock*, *paper*, or *scissors*. Then, both players reveal their choices. The outcome is determined as follows. If both players choose the same strategy, neither player wins or loses anything. Otherwise:

   - "paper covers rock": if one player chooses paper and the other chooses rock, the player who chose paper wins and is paid 1 by the other player.
   - "scissors cut paper": if one player chooses scissors and the other chooses paper, the player who chose scissors wins and is paid 1 by the other player.
   - "rock breaks scissors": if one player chooses rock and the other player chooses scissors, the player who chose rock wins and is paid 1 by the other player.

   We can write the payoff matrix for this game as follows:

	rock	paper	scissors
rock	0	-1	1
paper	1	0	-1
scissors	-1	1	0

   (a) Show that this game does not have a pure Nash equilibrium.

   (b) Show that the pair of mixed strategies $\mathbf{x}^\mathsf{T} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $\mathbf{y}^\mathsf{T} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ together are a Nash equilibrium.

   Solution:

   (a) If we compute the security level for each of Rosemary's strategies ("rock", "paper", and "scissors", respectively), we find $-1, -1$, and -1. If we compute the security level for each of Colin's strategies, we find $1, 1$, and 1. There is thus no pair of strategies where the security levels match and so there is no pure Nash equilibrium for the game (alternatively, we could note that the best security level for Rosemary is -1 and the best for Colin is 1, and these values are not the same).

   (b) We can compute the security level for $\mathbf{x}$ by considering each of Colin's pure strategies.

   - When he selects "rock", Rosemary's expected payoff for $\mathbf{x}$ will be: $0 + 1/3 - 1/3 = 0$.
   - When he selects "paper", Rosemary's expected payoff for $\mathbf{x}$ will be: $-1/3 + 0 + 1/3 = 0$.
   - When he selects "scissors", Rosemary's expected payoff for $\mathbf{x}$ will be: $1/3 - 1/3 + 0 = 0$.

   The smallest of these is 0, so this is the security level of $\mathbf{x}$.

   We now compute the security level for $\mathbf{y}$, by considering each of Rosemary's strategies:

   - When she selects "rock", the expected payoff for $\mathbf{y}$ will be: $0 - 1/3 + 1/3 = 0$.

- When she selects "paper", the expected payoff for $\mathbf{y}$ will be: $1/3+0-1/3 = 0$.
- When she selects "scissors", the expected payoff for $\mathbf{y}$ will be: $-1/3 + 1/3 + 0 = 0$.

The largest of these is 0, so this is the security level of $\mathbf{y}$. Since this is the same as the security level for $\mathbf{x}$, the strategies $\mathbf{x}^\mathsf{T} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $\mathbf{y}^\mathsf{T} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ together form a Nash equilibrium.

(c)

2. Suppose now we alter the game so that whenever Colin chooses "paper" the loser pays the winner 3 instead of 1:

	rock	paper	scissors
rock	0	-3	1
paper	1	0	-1
scissors	-1	3	0

(a) Show that $\mathbf{x}^\mathsf{T} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $\mathbf{y}^\mathsf{T} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ together are *not* a Nash equilibrium for this modified game.

(b) Formulate a linear program that can be used to calculate a mixed strategy $\mathbf{x} \in \Delta(R)$ that maximises Rosemary's security level for this modified game.

(c) Show that $\mathbf{x}^\mathsf{T} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $\mathbf{y}^\mathsf{T} = (\frac{3}{7}, \frac{1}{7}, \frac{3}{7})$ together are a Nash equilibrium for this game.

Solution:

(a) We can compute the security level for $\mathbf{x}$ by considering each of Colin's pure strategies.

- When he selects "rock", Rosemary's expected payoff for $\mathbf{x}$ will be: $0 + 1/3 - 1/3 = 0$.
- When he selects "paper", Rosemary's expected payoff for $\mathbf{x}$ will be: $-1 + 0 + 1 = 0$.
- When he selects "scissors", Rosemary's expected payoff for $\mathbf{x}$ will be: $1/3 - 1/3 + 0 = 0$.

The smallest of these is 0, so this is the security level of $\mathbf{x}$.

We now compute the security level for $\mathbf{y}$, by considering each of Rosemary's strategies:

- When she selects "rock", the expected payoff for $\mathbf{y}$ will be: $0 - 1 + 1/3 = -2/3$.
- When she selects "paper", the expected payoff for $\mathbf{y}$ will be: $1/3+0-1/3 = 0$.
- When she selects "scissors", the expected payoff for $\mathbf{y}$ will be: $-1/3+1+0 = 2/3$.

The largest of these is $2/3$, so this is the security level of $\mathbf{y}$. Since this is *not* the same as the security level for $\mathbf{x}$, the strategies $\mathbf{x}^\mathsf{T} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $\mathbf{y}^\mathsf{T} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ do not together form a Nash equilibrium.

(b) The linear program is given by:

$$\begin{aligned}
\text{maximize} \quad & x_4 \\
\text{subject to} \quad & x_4 \leq x_2 - x_3, \\
& x_4 \leq -3x_1 + 3x_3, \\
& x_4 \leq x_1 - x_2, \\
& x_1 + x_2 + x_3 = 1, \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}$$

we can rearrange this to obtain:

$$\begin{aligned}
\text{maximize} \quad & x_4 \\
\text{subject to} \quad & -x_2 + x_3 + x_4 \leq 0, \\
& 3x_1 - 3x_3 + x_4 \leq 0, \\
& -x_1 + x_2 + x_4 \leq 0, \\
& x_1 + x_2 + x_3 = 1, \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}$$

(c) We can compute the security level for $\mathbf{x}$ by considering each of Colin's pure strategies.

- When he selects "rock", Rosemary's expected payoff for $\mathbf{x}$ will be: $0 + 1/3 - 1/3 = 0$.
- When he selects "paper", Rosemary's expected payoff for $\mathbf{x}$ will be: $-1 + 0 + 1 = 0$.
- When he selects "scissors", Rosemary's expected payoff for $\mathbf{x}$ will be: $1/3 - 1/3 + 0 = 0$.

The smallest of these is 0, so this is the security level of $\mathbf{x}$.

We now compute the security level for $\mathbf{y}$, by considering each of Rosemary's strategies:

- When she selects "rock", the expected payoff for $\mathbf{y}$ will be: $0 - 3/7 + 3/7 = 0$.
- When she selects "paper", the expected payoff for $\mathbf{y}$ will be: $3/7 + 0 - 3/7 = 0$.
- When she selects "scissors", the expected payoff for $\mathbf{y}$ will be: $-3/7 + 3/7 + 0 = 0$.

The largest of these is 0, so this is the security level of $\mathbf{y}$. Since this is the same as the security level for $\mathbf{x}$, the strategies $\mathbf{x}^{\mathsf{T}} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $\mathbf{y}^{\mathsf{T}} = (\frac{3}{7}, \frac{1}{7}, \frac{3}{7})$ together form a Nash equilibrium.

3. You will be able to do this question after Thursday's lecture in Week 12. Suppose that we further alter the game from question 2 as follows: now whenever both players select the same strategy, *both* receive a payoff of 2. Note that this is no longer a zero-sum game.

(a) Give the payoff matrix for this game. As usual, you should list Rosemary's payoffs first and Colin's payoffs second in each cell.

Solution:

	rock	paper	scissors
rock	$(2,2)$	$(-3,3)$	$(1,-1)$
paper	$(1,-1)$	$(2,2)$	$(-1,1)$
scissors	$(-1,1)$	$(3,-3)$	$(2,2)$

(b) Underline the best responses for each player to each of the other players' strategies in your payoff matrix. Then, find and give all Pure Nash equilibria for the modified game.

Solution: Underlining the best responses for each player, we find:

	rock	paper	scissors
rock	$(\underline{2},2)$	$(-3,\underline{3})$	$(1,-1)$
paper	$(1,-1)$	$(2,\underline{2})$	$(-1,1)$
scissors	$(-1,1)$	$(\underline{3},-3)$	$(\underline{2},\underline{2})$

The only Pure Nash equilibrium is the pair of strategies ("scissors","scissors"), since this is the only pair of strategies that simultaneously a best response for both players.

4. Consider a 2-player zero-sum game with the following payoff matrix:

	c_1	c_2	c_3
r_1	1	2	−2
r_2	−13	4	12
r_3	1	−7	9

Give a linear program for computing Rosemary's best strategy (that is, the one that gives her the best security level). Also give a linear program for computing Colin's best strategy.

Solution: The linear program for Rosemary is

$$\begin{aligned}
\text{maximize} \quad & s \\
\text{subject to} \quad & s \leq x_1 - 13x_2 + x_3, \\
& s \leq 2x_1 + 4x_2 - 7x_3, \\
& s \leq -2x_1 + 12x_2 + 9x_3, \\
& x_1 + x_2 + x_3 = 1, \\
& x_1, x_2, x_3 \geq 0, \\
& s \text{ unrestricted}
\end{aligned}$$

The linear program for Colin is

$$\begin{aligned}
\text{minimize} \quad & t \\
\text{subject to} \quad & t \geq y_1 + 2y_2 - 2y_3, \\
& t \geq -13y_1 + 4y_2 + 12y_3, \\
& t \geq y_1 - 7y_2 + 9y_3, \\
& y_1 + y_2 + y_3 = 1, \\
& y_1, y_2, y_3 \geq 0, \\
& t \text{ unrestricted}
\end{aligned}$$