Main Examination period – May/June – Semester B

MTH5105: Differential and Integral Analysis-SAMPLE

Examiners:

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You will have a period of **3 hours** to complete the exam and submit your solutions.

You should attempt ALL questions. Marks available are shown next to the questions.

The exam is closed-book, and **no outside notes are allowed**.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiners:

Question 1 [25 marks].

(a) Let $f : (a, b) \to \mathbb{R}$ be a real valued function. State the definition for f to be **differentiable** at a point $x \in (a, b)$. [**5**]

[book]

Let $x \in (a, b), f : (a, b) \to \mathbb{R}$. The derivative of f at x is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{x \to x} \frac{f(x) - f(x)}{x - x}.$$

If this limit exists then f is differentiable at x.

(b) Consider the following function, $g : (0, \infty) \to \mathbb{R}$ given by

$$g(x) = \sqrt{x}.$$

Using the definition of derivative, compute the derivative of g. [**5**]

[book]

$$\begin{aligned}
\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} &= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\
&= \lim_{h \to 0} \frac{x + h - x}{h(\sqrt{x+h} + \sqrt{x})}, \quad \text{as } h \neq 0 \\
&= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.
\end{aligned}$$

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right), & x > 0, \\ 0, & x \leq 0. \end{cases}$$

Is f differentiable? (Fully explain your answer) [**5**]

[unseen]

We have

$$\frac{f(x) - f(0)}{x - 0} = x \sin \frac{1}{x^2}.$$

Note that

$$\left| \sin \frac{1}{x^2} \right| \leq 1$$

which gives

$$\left|\lim_{x\to 0} x\sin\frac{1}{x^2}\right| = \lim_{x\to 0}\left|x\sin\frac{1}{x^2}\right| \le \lim_{x\to 0}|x| = 0$$

hence

$$f'(0) = \lim_{x\to 0}\left(x\sin\frac{1}{x^2}\right) = 0.$$

(d) State the **Mean Value Theorem**. [**5**]

[book]

Suppose that $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). Then there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(e) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Show that if $|f'(c)| \le M$ for all $c \in \mathbb{R}$ then for all $x, y \in \mathbb{R}$ we have

$$|f(x) - f(y)| \le M|x - y|.$$ [**5**]

Without loss of generality, assume that $y > x$. We then apply the Mean Value Theorem on $[x,y]$. We obtain

$$f(y) - f(x) = f'(\xi)(y - x)$$

for some $\xi \in (x,y)$. This implies that

$$|f(x) - f(y)| \le |f'(\xi)||y - x| \le M|y - x|.$$

[book]

Question 2 [25 marks].

(a) State the definition of a **uniformly continuous function**. [**5**]

[book]

Suppose that $f : I \to \mathbb{R}$ where I is an interval. We say that f is <u>uniformly continuous</u> on Ω if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon)$ such that for all $x, y \in I$ where $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$. More compactly this means

$$\forall\varepsilon > 0 \exists\delta > 0 \forall x, y \in \Omega \mid |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

(b) Prove that $f(x) = \frac{1}{x}$ is uniformly continuous on $[a, 2], 0 < a < 2$. [**5**]

Let $\varepsilon > 0$ be given. Now consider $x, y \in [a, 2]$. It follows that

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|x-y|}{|x||y|} \leq \frac{|x-y|}{a^2}$$

as $|x|, |y| \geq a$. Hence if we choose $\delta = a^2\varepsilon$ then $|x - y| < \delta = a^2\varepsilon$ implies

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| < \varepsilon.$$

Alternatively, the function $\frac{1}{x}$ is a continuous function on a closed bounded interval and hence uniformly continuous.

--- [book]

(c) Let $f_n(x) = \frac{1}{n}x^{n^2}, \quad x \in [-1, 1]$.

(i) For each $x \in [-1, 1]$ compute $\lim_{n\to\infty} f_n(x)$. [**5**]

--- [unseen]

Note that

$$\lim_{n\to\infty} |f_n(x)| = \lim_{n\to\infty} \frac{|x^{n^2}|}{n} \leq \lim_{n\to\infty} \frac{1}{n} = 0.$$

Hence we have that $f(x) = 0$.

(ii) For each $x \in [-1, 1]$ Let $f(x) = \lim_{n\to\infty} f_n(x)$. Does f_n converge to f uniformly on $[-1, 1]$? Justify your answer. [**5**]

--- [unseen]

Yes. We have

$$|f_n(x) - f(x)| = |f_n(x)| = \frac{|x|^{n^2}}{n} \leq \frac{1}{n}.$$

Hence we we choose $\varepsilon > \frac{1}{n}$ or $n > \frac{1}{\varepsilon}$ where n is independent of x, we see that f_n converges uniformly to $f \equiv 0$.

(iii) Show that the following limit exists and compute its value,

$$\lim_{n\to\infty} \int_{-1}^{1} f_n(x)dx.$$

[**5**]

--- [unseen]

As f_n converges uniformly to f and each f_n is continuous, we may interchange the integral and limit to get that

$$\lim_{n\to\infty}\int_{-1}^{1} f_n(x)dx = \int_{-1}^{1}\lim_{n\to\infty} f_n(x)dx = \int_{-1}^{1} 0dx = 0.$$

Question 3 [25 marks].

(a) State the **Inverse Function Theorem**. [**5**]

[book]

Let f be a one-to-one continuous function on an open interval I and let $J = f(I)$. If f is differentiable at $x_0 \in I$ and if $f'(x_0) \neq 0$ then f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

(b) Let $f(x) = \exp(x), x \in \mathbb{R}$. Show that f is invertible and if $g(y) = f^{-1}(y)$ is the inverse of f, compute the derivative of $f^{-1}(y)$ in terms of y. [**5**]

[unseen]

Since $f'(x) = \exp(x) > 0$ it follows that f is strictly increasing and therefore f is injective. We find that if $y = f(x) = \exp(x)$ then by the inverse function theorem

$$(f^{-1})'(y) = \frac{1}{f'(x))} = \frac{1}{\exp(x)} = \frac{1}{y}.$$

(c) Let $h : (-1, 1) \to \mathbb{R}$ be the function given by

$$h(x) = \frac{1}{1+x}.$$

Using any correct method, compute the Taylor series of h about $x = 0$ together with its interval of convergence. [**7**]

[book]

The series is the geometric series for $|x| < 1$ and the Taylor series coincides with its power series expansion hence

$$\sum_{n=0}^{\infty} (-x)^n = \frac{1}{1+x}.$$

Alternative method is simply to compute the derivatives of h to n-th order. In this case, proving by induction,

$$\frac{d}{dx} h(x) = (-1)^n \frac{n!}{(1+x)^{n+1}}$$

Hence we have that

$$\left.\frac{d^n}{dx^n} h(x)\right|_{x=0} = (-1)^n n!$$

Therefore the Taylor series about $x = 0$ is given by

$$
\begin{aligned}
Th(x) &= \sum_{k=0} \frac{f^{(k)}(0)}{k!} x^k \\
&= \sum_{k=0}^{\infty} (-x)^n.
\end{aligned}
$$

The radius of convergence is then given by

$$R = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1.$$

Therefore $R = 1$. At $x = 1$ and $x = -1$ the series diverges, hence the interval of convergence in $(-1, 1)$

(d) Compute the antiderivatives of h. **[2]**

The antiderivatives if h are given by $\log(|x + 1|) + c$.

(e) Using part (d) above give a Taylor expansion for $\log(1 + x)$ about $x = 0$ together with its interval of convergence. **[6]**

[unseen]

Since $\sum_{n=0}^{\infty}(-x)^n$ converges uniformly to $\frac{1}{1+x}$ on its interval of convergence, we may integrate term by term

$$
\begin{aligned}
\log(1+x) &= \int_0^x \frac{1}{1+t} dt \\
&= \int_0^x \sum_{n=0}^{\infty} (-1)^n t^n dt \\
&= \sum_{n=0}^{\infty} (-1)^n \int_0^x t^n dt \\
&= \sum_{n=0}^{\infty} \left[\frac{1}{n+1} t^{n+1} \right]_0^x \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}.
\end{aligned}
$$

The radius of convergence is the same as for h so $R = 1$. At $x = 1$ the series converges as it is an alternating harmonic series and at $x = -1$ the series diverges as it is a harmonic series. Hence the interval of convergence is $(-1, 1]$.

Question 4 [20 marks].

(a) State the **Mean Value Theorem for Integrals**. **[5]**

[book]

Let f be a continuous function on $[a,b]$ then there exists $c \in [a,b]$ such that

$$\int_a^b f(x)dx = f(c)(b-a).$$

(b) Consider the function $g : [0,1] \to \mathbb{R}, g(x) = x$.

(i) Show that g is Riemann integrable. **[2]**

[book]

g is a continuous function on a closed bounded interval $[0,1]$ and hence is Riemann integrable.

(ii) Show that the upper sum $U(g, P_n)$ of g for the equidistant partition

$$P_n = \left\{ x_0 = 0, \cdots, x_k = \frac{k}{n}, \cdots, x_n = 1 \right\}.$$ **[6]**

satisfies $\lim_{n\to\infty} U(g, P_n) = \frac{1}{2}$.

$\big($*You may use the formula,* $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, *or any other correct method.*$\big)$

We find that as $\triangle = \frac{1}{n}$

$$M_k = \sup_{[\frac{k-1}{n}, \frac{k}{n}]} g(x) = \frac{k}{n}.$$

Therefore

$$\begin{aligned} U(g, P_n) &= \sum_{k=1}^{n} M_k (x_k - x_{k-1}) \\ &= \sum_{k=1}^{n} \frac{k}{n^2} \\ &= \frac{n(n+1)}{2n^2}. \end{aligned}$$

Therefore

$$\lim_{n\to\infty} U(g, P_n) = \lim_{n\to\infty} \frac{n(n+1)}{2n^2} = \frac{1}{2}.$$

[seen]

(iii) Using part (i) and (ii) compute the integral $\int_0^1 g(x)dx$. [**2**]

Since g is Riemann integrable we have $\int_0^1 g(x)dx = \lim_{n\to\infty} U(g, P_n) = \frac{1}{2}$.

[unseen]

For the remainder of this question, let $f : [a, b] \to \mathbb{R}$ denote a continuous function.

(c) Let F, G be antiderivatives of f. What is the relation between F and G?

[book]

If $G'(x) = f(x)$ on (a, b) then we have $G' - F' = (G - F)' = f - f = 0$ so that $G - F = c$ so $G = F + c$, where $c \in \mathbb{R}$. Hence F and G differ by a constant. Also this gives us

$$\int_a^b f(x)dx = G(b) - G(a) = F(b) + c - F(a) - c = F(b) - F(a).$$

(d) Let $f : \mathbb{R} \to \mathbb{R}$ and denote by H the following function,

$$H(x) = \int_{x-1}^{x+1} f(t)dt.$$

Show that H is differentiable and find its derivative. [**5**]

[unseen]

Since f is continuous on $\mathbb{R}$ we let F be an antiderivative of f. This implies that

$$\int_{x-1}^{x+1} h(t)dt = F(x + 1) - F(x - 1) = H(x).$$

As F is a differentiable function, so is H. Finally computing the derivative using the chain rule, we get

$$\begin{aligned} H'(x) &= F'(x + 1) - F'(x - 1) \\ &= f(x + 1) - f(x - 1). \end{aligned}$$

End of Paper.