3 hours

Main Examination period – May/June – Semester B

MTH5105: Differential and Integral Analysis

Examiners:

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You will have a period of **3 hours** to complete the exam and submit your solutions.

You should attempt ALL questions. Marks available are shown next to the questions.

The exam is closed-book, and **no outside notes are allowed**.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

Possession of unauthorised material at any time when under examination conditions is an assessment offence and can lead to expulsion from QMUL. Check now to ensure you do not have any unauthorised notes, mobile phones, smartwatches or unauthorised electronic devices on your person. If you do, raise your hand and give them to an invigilator immediately.

It is also an offence to have any writing of any kind on your person, including on your body. If you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms, it will be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. A mobile phone that causes a disruption in the exam is also an assessment offence.

Exam papers must not be removed from the examination room.

Examiners:

Question 1 [25 marks].

(a) Let $f : (a, b) \to \mathbb{R}$ be a real valued function. State the definition for f to be **differentiable** at a point $x \in (a, b)$. [5]

(b) Consider the following function, $g : (0, \infty) \to \mathbb{R}$ given by

$$g(x) = \sqrt{x}.$$

Using the definition of derivative, compute the derivative of g. [5]

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right), & x > 0, \\ 0, & x \le 0. \end{cases}$$

Find where f is differentiable? (Fully explain your answer.) [5]

(d) State the **Mean Value Theorem**. [5]

(e) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Show that if $|f'(c)| \le M$ for all $c \in \mathbb{R}$ then for all $x, y \in \mathbb{R}$ we have

$$|f(x) - f(y)| \le M|x - y|.$$

[5]

Question 2 [25 marks].

(a) State the definition of a **uniformly continuous function**. [5]

(b) Prove that $f(x) = \frac{1}{x}$ is uniformly continuous on $[a, 2], 0 < a < 2$. [5]

(c) Let $f_n(x) = \frac{1}{n} x^{n^2}, \quad x \in [-1, 1]$.

(i) For each $x \in [-1, 1]$ compute $\lim_{n\to\infty} f_n(x)$. [5]

(ii) For each $x \in [-1, 1]$ let $f(x) = \lim_{n\to\infty} f_n(x)$. Does f_n converge to f uniformly on $[-1, 1]$? Justify your answer. [5]

(iii) Show that the following limit exists and compute its value,

$$\lim_{n\to\infty} \int_{-1}^{1} f_n(x)dx.$$

[5]

Question 3 [25 marks].

(a) State the **Inverse Function Theorem**. [5]

(b) Let $f(x) = \exp(x), x \in \mathbb{R}$. Show that f is invertible and compute the derivative of $f^{-1}(y)$ in terms of y. [5]

(c) Let $h : \mathbb{R}\backslash\{-1\} \to \mathbb{R}$ be the function given by

$$h(x) = \frac{1}{1+x}.$$

Using any correct method, compute the Taylor series of h about $x = 0$ together with its interval of convergence. [7]

(d) Compute the antiderivatives of h defined above. [2]

(e) Using part (d) above give a Taylor expansion for $\log(1 + x)$ about $x = 0$ together with its interval of convergence. [6]

Question 4 [25 marks].

(a) State the **Mean Value Theorem for Integrals**. [5]

(b) Consider the function $g : [0, 1] \to \mathbb{R}, g(x) = x$.

(i) Show that g is Riemann integrable. [2]

(ii) Compute the upper sum $U(g, P_n)$ of g for the equidistant partition

$$P_n = \left\{x_0 = 0, \cdots, x_k = \frac{k}{n}, \cdots, x_n = 1\right\}.$$ [6]

(*You may use the formula,* $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, *or any other correct method.*)

(iii) Using part (i) and (ii) compute the integral $\int_0^1 g(x)dx$. [2]

(c) Let $f : [a, b] \to \mathbb{R}$ denote a bounded function. Suppose F, G are antiderivatives of f. What is the relation between F and G? [5]

(d) Let $f : \mathbb{R} \to \mathbb{R}$ and denote by H the following function,

$$H(x) = \int_{x-1}^{x+1} f(t)dt.$$

Show that H is differentiable and find its derivative. [5]

End of Paper.