MTH5113 (2022/23): Problem Sheet 11
Solutions

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(1) *(Boring computations—but you should know how to do them)* Consider the following vector fields $\mathbf{F}$, $\mathbf{G}$, $\mathbf{H}$ on $\mathbb{R}^3$, where:

$$\begin{aligned}
\mathbf{F}(x, y, z) &= (x,\ y,\ z)_{(x,y,z)}, \\
\mathbf{G}(x, y, z) &= (x^2,\ -2xy,\ 3xz)_{(x,y,z)}, \\
\mathbf{H}(x, y, z) &= (x^2 + y^2 + z^2,\ x^4 - y^2z^2,\ xyz)_{(x,y,z)}.
\end{aligned}$$

(a) Compute the divergence of $\mathbf{F}$, $\mathbf{G}$, and $\mathbf{H}$ at each point $(x, y, z) \in \mathbb{R}^3$.

(b) Compute the curl of $\mathbf{F}$, $\mathbf{G}$, and $\mathbf{H}$ at each point $(x, y, z) \in \mathbb{R}^3$.

(a) For $\mathbf{F}$, we have, by the definition of the divergence,

$$(\nabla \cdot \mathbf{F})(x, y, z) = \partial_x(x) + \partial_y(y) + \partial_z(z) = 3.$$

Similarly, for $\mathbf{G}$, we have

$$\begin{aligned}
(\nabla \cdot \mathbf{G})(x, y, z) &= \partial_x(x^2) + \partial_y(-2xy) + \partial_z(3xz) \\
&= 2x - 2x + 3x \\
&= 3x.
\end{aligned}$$

Finally, for $\mathbf{H}$,

$$\begin{aligned}
(\nabla \cdot \mathbf{H})(x, y, z) &= \partial_x(x^2 + y^2 + z^2) + \partial_y(x^4 - y^2z^2) + \partial_z(xyz) \\
&= 2x - 2yz^2 + xy.
\end{aligned}$$

(b) For $\mathbf{F}$, we have, by the definition of the curl,

$$\begin{aligned}
(\nabla \times \mathbf{F})(x, y, z) &= (\partial_y(z) - \partial_z(y),\ \partial_z(x) - \partial_x(z),\ \partial_x(y) - \partial_y(x))_{(x,y,z)} \\
&= (0, 0, 0)_{(x,y,z)}.
\end{aligned}$$

Similarly, for **G**, we have

$$\begin{aligned}(\nabla \times \mathbf{G})(x,y,z) &= (\partial_y(3xz) - \partial_z(-2xy),\ \partial_z(x^2) - \partial_x(3xz),\ \partial_x(-2xy) - \partial_y(x^2))_{(x,y,z)} \\ &= (0,\ -3z,\ -2y)_{(x,y,z)}.\end{aligned}$$

Lastly, the curl of **H** satisfies

$$(\nabla \times \mathbf{H})(x,y,z) = (A_1(x,y,z),\ A_2(x,y,z),\ A_3(x,y,z))_{(x,y,z)},$$

where

$$\begin{aligned}A_1(x,y,z) &= \partial_y(xyz) - \partial_z(x^4 - y^2z^2) = xz + 2y^2z, \\ A_2(x,y,z) &= \partial_z(x^2+y^2+z^2) - \partial_x(xyz) = 2z - yz, \\ A_3(x,y,z) &= \partial_x(x^4 - y^2z^2) - \partial_y(x^2+y^2+z^2) = 4x^3 - 2y.\end{aligned}$$

As a result, we conclude that

$$(\nabla \times \mathbf{H})(x,y,z) = (xz + 2y^2z,\ 2z - yz,\ 4x^3 - 2y)_{(x,y,z)}.$$

(2) *(Fun with Green's theorem)* Let C denote the circle

$$C = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 9\},$$

and let us assign to C the anticlockwise orientation.

(a) Let **F** be the vector field on $\mathbb{R}^2$ defined by

$$\mathbf{F}(x,y) = (x - y,\ x + y)_{(x,y)}.$$

Compute the curve integral of **F** over C *directly.*

(b) Compute the curve integral from part (a) *using Green's theorem.* Check that your answer here matches the answer you obtained in part (a).

(c) Let **G** be the vector field on $\mathbb{R}^2$ defined by

$$\mathbf{G}(x,y) = (x^{9999999999999}e^x + y,\ x + y^{33333333333333}e^{2y})_{(x,y)}.$$

Compute (using your favourite method) the curve integral of $\mathbf{G}$ over C.

(a) To compute the curve integral, we first parametrise C:

$$\gamma : (0, 2\pi) \to \mathbb{R}^2, \qquad \gamma(t) = (3\cos t,\ 3\sin t).$$

Note that γ is an injective parametrisation of C, whose image is all of C except for a single point, and which generates the anticlockwise orientation of C. Thus, we have

$$\int_C \mathbf{F} \cdot d\mathbf{s} = + \int_0^{2\pi} [\mathbf{F}(\gamma(t)) \cdot \gamma'(t)_{\gamma(t)}]\, dt.$$

To compute this integral, we observe that

$$\begin{aligned} \gamma'(t) &= (-3\sin t,\ 3\cos t), \\ \mathbf{F}(\gamma(t)) &= (3\cos t - 3\sin t,\ 3\cos t + 3\sin t)_{(3\cos t, 3\sin t)}, \\ \mathbf{F}(\gamma(t)) \cdot \gamma'(t)_{\gamma(t)} &= 9\sin^2 t + 9\sin^2 t = 9. \end{aligned}$$

As a result,

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} 9\, dt = 18\pi.$$

(b) Let F_1 and F_2 denote the directional components of $\mathbf{F}$:

$$F_1(x, y) = x - y, \qquad F_2(x, y) = x + y, \qquad (x, y) \in \mathbb{R}^2.$$

Note in particular that

$$\partial_1 F_2(x, y) - \partial_2 F_1(x, y) = 1 - (-1) = 2.$$

Thus, by Green's theorem, we have that

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \iint_D [\partial_1 F_2(x, y) - \partial_2 F_1(x, y)]\, dx dy = \iint_D 2\, dx dy,$$

where D is the interior of the circle C:

$$D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 9\}.$$

(Note in particular that the anticlockwise orientation of C is the "positive" orientation obtained from Green's theorem.) Finally, since D is simply the circular disk of radius 3, then

$$\int_C \mathbf{F} \cdot d\mathbf{s} = 2 \cdot \mathcal{A}(D) = 2 \cdot (\pi \cdot 3^2) = 18\pi.$$

(c) Let G_1 and G_2 denote the directional components of $\mathbf{G}$:

$$G_1(x, y) = x^{9999999999999} e^x + y, \qquad G_2(x, y) = x + y^{3333333333333} e^{2y}.$$

Note that

$$\partial_1 G_2(x, y) - \partial_2 G_1(x, y) = 1 - 1 = 0.$$

Thus, applying Green's theorem, and letting D be as in part (b), we have

$$\int_C \mathbf{G} \cdot d\mathbf{s} = \iint_D [\partial_1 G_2(x, y) - \partial_2 G_1(x, y)] \, dx dy = 0.$$

(3) *(Fun with Stokes' theorem)* Let S denote the upper half-sphere,

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1, \, z > 0\}.$$

In addition, let C denote the boundary of S, i.e. the circle

$$C = \{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\},$$

and assign to C the (anticlockwise) orientation generated by the parametrisation

$$\gamma : \mathbb{R} \to \mathbb{R}^3, \qquad \gamma(t) = (\cos t, \, \sin t, \, 0).$$

(a) Let $\mathbf{F}$ be the vector field on $\mathbb{R}^3$ given by

$$\mathbf{F}(x, y, z) = (x, \, y, \, z)_{(x,y,z)}.$$

Compute directly the curve integral of $\mathbf{F}$ over C.

(b) Evaluate the integral from part (a) by applying *Stokes' theorem* and computing instead an appropriate integral over S. Make sure that you obtain the same answer as in (a).

(c) Let $\mathbf{G}$ and $\mathbf{H}$ be smooth vector fields on $\mathbb{R}^3$, and assume that $\mathbf{G}(\mathbf{p}) = \mathbf{H}(\mathbf{p})$ for every

$\mathbf{p} \in C$. Using Stokes' theorem, show that

$$\iint\limits_S (\nabla \times \mathbf{G}) \cdot d\mathbf{A} = \iint\limits_S (\nabla \times \mathbf{H}) \cdot d\mathbf{A}.$$

(Here, S can be assigned either of its orientations.)

(a) First, note that the function

$$\gamma_* : (0, 2\pi) \to C, \qquad \gamma_*(t) = (\cos t,\ \sin t,\ 0)$$

is an injective parametrisation of C that covers all of C except for a single point $(1, 0, 0)$. (Note also that γ_* generates our given orientation of C.) As a result,

$$\int_C \mathbf{F} \cdot d\mathbf{s} = +\int_0^{2\pi} \left[\mathbf{F}(\gamma_*(t)) \cdot \gamma_*'(t)_{\gamma_*(t)} \right] dt.$$

To compute the above, we notice that

$$\begin{aligned} \mathbf{F}(\gamma_*(t)) &= (\cos t,\ \sin t,\ 0)_{(\cos t, \sin t, 0)}, \\ \gamma_*'(t) &= (-\sin t,\ \cos t,\ 0), \\ \mathbf{F}(\gamma_*(t)) \cdot \gamma_*'(t)_{\gamma_*(t)} &= 0. \end{aligned}$$

As a result, we obtain our solution,

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} 0 \, dt = 0.$$

(b) We begin by applying Stokes' theorem to obtain

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \iint\limits_S (\nabla \times \mathbf{F}) \cdot d\mathbf{A},$$

where S is assigned the upward-facing orientation (i.e. given by the side of S facing the positive z-direction). In particular, this orientation of S is associated, via Stokes' theorem, to our given orientation of C; see the examples in the final lectures.

Next, observe that by a direct computation, we have

$$(\nabla \times \mathbf{F})(x, y, z) = (\partial_y z - \partial_z y,\ \partial_z x - \partial_x z,\ \partial_x y - \partial_y x)_{(x,y,z)} = (0, 0, 0)_{(x,y,z)}.$$

Since $\nabla \times \mathbf{F}$ vanishes everywhere, it follows that

$$\int_C \mathbf{F} \cdot \mathbf{ds} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{dA} = 0.$$

(c) Since $\mathbf{G}$ and $\mathbf{H}$ are assumed to be identical on C, we trivially have

$$\int_C \mathbf{G} \cdot \mathbf{ds} = \int_C \mathbf{H} \cdot \mathbf{ds}.$$

(Here, we can use either orientation of C, whichever matches our choice of orientation for S.) Thus, applying Stokes' theorem twice, we obtain

$$\iint_S (\nabla \times \mathbf{G}) \cdot \mathbf{dA} = \int_C \mathbf{G} \cdot \mathbf{ds} = \int_C \mathbf{H} \cdot \mathbf{ds} = \iint_S (\nabla \times \mathbf{H}) \cdot \mathbf{dA}.$$

(4) *(Fun with the divergence theorem)* Let $\mathbb{S}^2$ denote the unit sphere centred at the origin,

$$\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\},$$

and assign to $\mathbb{S}^2$ the outward-facing orientation.

(a) Let $\mathbf{F}$ be the vector field on $\mathbb{R}^3$ given by

$$\mathbf{F}(x, y, z) = (x,\ y,\ z)_{(x,y,z)}.$$

Compute directly the surface integral of $\mathbf{F}$ over $\mathbb{S}^2$.

(b) Evaluate the integral from part (a) by applying the *divergence theorem* and computing an appropriate triple integral. Make sure that you obtain the same answer as in (a).

(c) Let $\mathbf{L}$ be the vector field on $\mathbb{R}^3$ given by

$$\mathbf{L}(x, y, z) = \left(y^{543} e^{y^2+z^4} z^{5234},\ e^{x^{562} z^{27} - x^{12} z^{43}} (x + z e^x)^{127},\ 1 + x^{10} y + 24 y^{17} e^{42 y^3} \right)_{(x,y,z)}.$$

Evaluate the surface integral of $\mathbf{L}$ over $\mathbb{S}^2$.

(a) To compute this surface integral, we first recall that

$$\rho_* : (0, 2\pi) \times (0, \pi) \to \mathbb{S}^2, \qquad \rho_*(u, v) = (\cos u \sin v,\ \sin u \sin v,\ \cos v)$$

is an injective parametrisation of $\mathbb{S}^2$ which covers "almost all" of $\mathbb{S}^2$, and that ρ_* generates the inward-facing orientation of $\mathbb{S}^2$. As a result, we have that

$$\iint_{\mathbb{S}^2} \mathbf{F} \cdot d\mathbf{A} = - \iint_{(0,2\pi)\times(0,\pi)} \left\{ \mathbf{F}(\rho_*(u, v)) \cdot [\partial_1 \rho_*(u, v) \times \partial_2 \rho_*(u, v)]_{\rho_*(u,v)} \right\} du dv$$

In addition, direct computations (include work from previous weeks) yield

$$\begin{aligned} \mathbf{F}(\rho_*(u, v)) &= \rho_*(u, v)_{\rho_*(u,v)}, \\ \partial_1 \rho_*(u, v) \times \partial_2 \rho(u, v) &= - \sin v \cdot \rho_*(u, v), \\ \mathbf{F}(\rho_*(u, v)) \cdot [\partial_1 \rho_*(u, v) \times \partial_2 \rho_*(u, v)]_{\rho_*(u,v)} &= - \sin v. \end{aligned}$$

Thus, combining all the above and applying Fubini's theorem, we conclude that

$$\begin{aligned} \iint_{\mathbb{S}^2} \mathbf{F} \cdot d\mathbf{A} &= - \int_0^{2\pi} \int_0^{\pi} (- \sin v) \, dv du \\ &= 2\pi \int_0^{\pi} \sin v \, dv \\ &= 4\pi. \end{aligned}$$

(b) Let W denote the region bounded by $\mathbb{S}^2$, namely, the ball of radius 1 centred at the origin. Moreover, observe that the divergence of $\mathbf{F}$ satisfies

$$(\nabla \cdot \mathbf{F})(x, y, z) = \partial_x x + \partial_y y + \partial_z z = 3.$$

Thus, by the divergence theorem, we obtain

$$\iint_{\mathbb{S}^2} \mathbf{F} \cdot d\mathbf{A} = \iiint_W (\nabla \cdot \mathbf{F})(x, y, z) \, dx dy dz = 3 \cdot \iiint_W dx dy dz.$$

Now, the integral of 1 over W is the volume of the unit ball W, which is simply

$$\frac{4\pi}{3} \cdot 1^3 = \frac{4\pi}{3}.$$

(Remember your pre-university mathematics!) As a result of this, we obtain

$$\iint_{\mathbb{S}^2} \mathbf{F} \cdot d\mathbf{A} = 3 \cdot \frac{4\pi}{3} = 4\pi.$$

(c) Note that the divergence of $\mathbf{L}$ vanishes everywhere:

$$\begin{aligned}(\nabla \cdot \mathbf{L})(x, y, z) &= \partial_x \left(y^{543} e^{y^2+z^4} z^{5234} \right) + \partial_y \left[e^{x^{562} z^{27} - x^{12} z^{43}} (x + ze^x)^{127} \right] \\ &\quad + \partial_z \left(1 + x^{10} y + 24 y^{17} e^{42y^3} \right) \\ &= 0.\end{aligned}$$

Thus, letting W be the region bounded by $\mathbb{S}^2$ and using the divergence theorem, we obtain

$$\iint_{\mathbb{S}^2} \mathbf{L} \cdot d\mathbf{A} = \iiint_W (\nabla \cdot \mathbf{L})(x, y, z) \, dx dy dz = \iiint_W 0 \, dx dy dz = 0.$$

(5) *(Setting boundaries I)*

(a) Describe the boundaries of the following subsets of $\mathbb{R}^2$, as one or more curves in $\mathbb{R}^2$:

(i) $D_1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + 4y^2 < 4\}$.

(ii) $D_2 = (0, 1) \times (0, 2)$.

(iii) $D_3 = \{(x, y) \in \mathbb{R}^2 \mid x < 1, \ y > 0, \ y < x\}$.

(b) Describe the boundaries of the following surfaces in $\mathbb{R}^3$, as one or more curves in $\mathbb{R}^3$:

(i) $S_1 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1, \ x > 0\}$.

(ii) $S_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, \ -1 < z < 1\}$.

(iii) $S_3 = \{(x, y, 0) \in \mathbb{R}^3 \mid x \in (0, 1), \ y \in (0, 2)\}$.

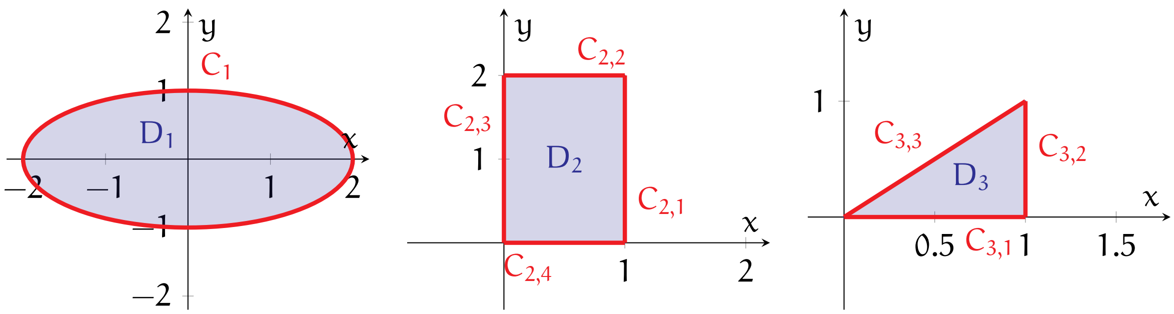
(c) Describe the boundaries of the following regions in $\mathbb{R}^3$, as one or more surfaces in $\mathbb{R}^3$:

(i) $V_1 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 < 9\}$.

(ii) $V_2 = (0, 1) \times (0, 1) \times (0, 1)$.

(iii) $V_3 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 < 1, \ -1 < z < 1\}$.

(a) Plots of D_1, D_2, D_3, and their boundaries are given below:



(i) Since D_1 is the inside of the ellipse $x^2 + 4y^2 = 4$, its boundary is simply the ellipse itself:

$$C_1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + 4y^2 = 4\}.$$

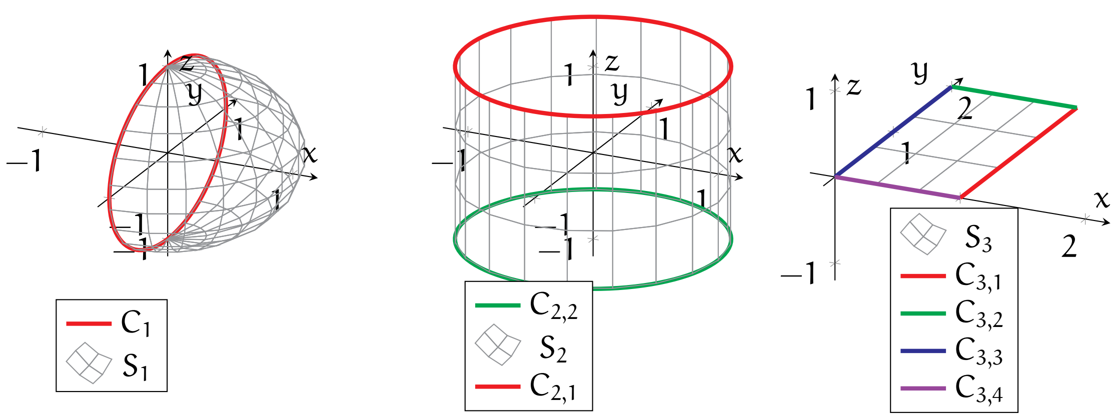
(ii) Here, D_2 is a rectangular region. Its boundary consists of the 4 sides of the rectangle, which can be expressed as 4 separate curves (or more specifically, line segments):

$$C_{2,1} = \{(1, y) \mid 0 < y < 2\}, \qquad C_{2,2} = \{(x, 2) \mid 0 < x < 1\},$$
$$C_{2,3} = \{(0, y) \mid 0 < y < 2\}, \qquad C_{2,4} = \{(x, 0) \mid 0 < x < 1\}.$$

(iii) Observe that D_3 is the triangle that bounded by the x-axis, the vertical line $x = 1$, and the diagonal line $y = x$. Thus, its boundary is simply the sides of this triangle:

$$C_{3,1} = \{(x, 0) \mid 0 < x < 1\}, \qquad C_{3,2} = \{(1, y) \mid 0 < y < 1\}, \qquad C_{3,3} = \{(x, x) \mid 0 < x < 1\}.$$

(b) Plots of S_1, S_2, S_3, and their boundaries are given below:



(i) Note that S_1 is the "right half" of the unit sphere $\mathbb{S}^2$ centred at the origin. Thus, its boundary is the unit circle that is the intersection of $\mathbb{S}^2$ and the yz-plane:

$$C_1 = \{(0, y, z) \in \mathbb{R}^3 \mid y^2 + z^2 = 1\}.$$

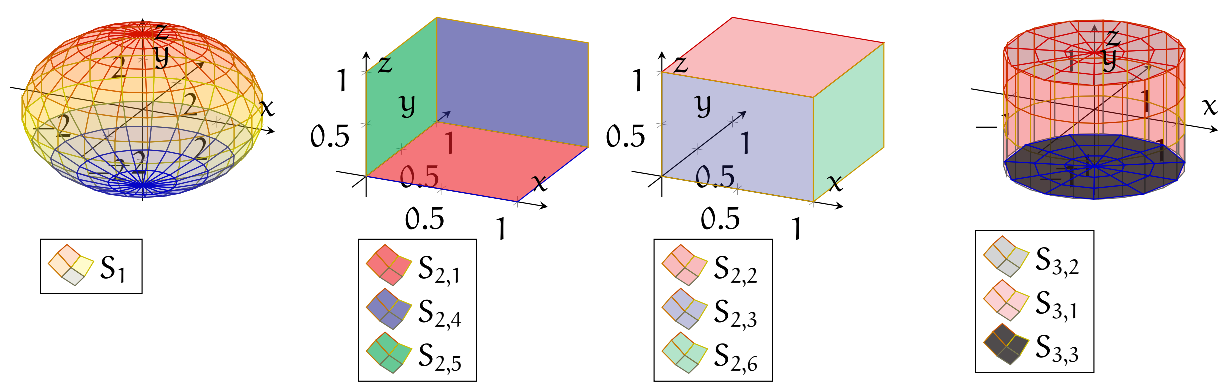
(ii) Observe that S_2 is the segment of the usual cylinder (centred about the z-axis) lying within the range $-1 < z < 1$. Thus, the boundary of S_2 consists of the cross-sections of the cylinder at $z = 1$ and $z = -1$, i.e. two separate circles:

$$C_{2,1} = \{(x, y, 1) \mid x^2 + y^2 = 1\}, \qquad C_{2,2} = \{(x, y, 1) \mid x^2 + y^2 = 1\}.$$

(iii) Here, S_3 is simply a rectangle lying on the xy-plane (in fact, the same rectangle as in part (a.ii)). Thus, the boundary of S_3 consists of the 4 sides of the rectangle:

$$C_{3,1} = \{(1, y, 0) \mid 0 < y < 2\}, \qquad C_{3,2} = \{(x, 2, 0) \mid 0 < x < 1\},$$
$$C_{3,3} = \{(0, y, 0) \mid 0 < y < 2\}, \qquad C_{3,4} = \{(x, 0, 0) \mid 0 < x < 1\}.$$

(c) The boundaries of V_1, V_2, V_3 are illustrated below (the regions V_1, V_2, V_3 themselves are are simply the volumes enclosed by the drawn surfaces):



(i) Since V_1 is the inside of the sphere $x^2 + y^2 + z^2 = 9$, its boundary is the sphere itself:

$$S_1 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 9\}.$$

(ii) V_2 is a unit cube. Thus, the boundary of V_2 is given by the 6 faces of this cube:

$$S_{2,1} = \{(x, y, 0) \in \mathbb{R}^3 \mid x, y \in (0, 1)\}, \qquad S_{2,2} = \{(x, y, 1) \in \mathbb{R}^3 \mid x, y \in (0, 1)\},$$
$$S_{2,3} = \{(x, 0, z) \in \mathbb{R}^3 \mid x, z \in (0, 1)\}, \qquad S_{2,4} = \{(x, 1, z) \in \mathbb{R}^3 \mid x, z \in (0, 1)\},$$
$$S_{2,5} = \{(0, y, z) \in \mathbb{R}^3 \mid y, z \in (0, 1)\}, \qquad S_{2,6} = \{(1, y, z) \in \mathbb{R}^3 \mid y, z \in (0, 1)\}.$$

(iii) V_3 is the interior of the usual cylinder (of radius 1, centred about the z-axis), restricted to the range $-1 < z < 1$. Thus, its boundary consists of the segment of cylinder satisfying $-1 < z < 1$, as well as the disks forming top and bottom "lids" of V_3:

$$S_{3,1} = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, \ -1 < z < 1\},$$
$$S_{3,2} = \{(x, y, 1) \in \mathbb{R}^3 \mid x^2 + y^2 < 1\},$$
$$S_{3,3} = \{(x, y, -1) \in \mathbb{R}^3 \mid x^2 + y^2 < 1\}.$$

(6) *(Setting boundaries II)*

(a) Suppose you apply Green's theorem to each of the regions D_i in Question (5a). Describe the resulting orientation(s) on the boundary of each D_i that you would obtain, according to the statement of Green's theorem.

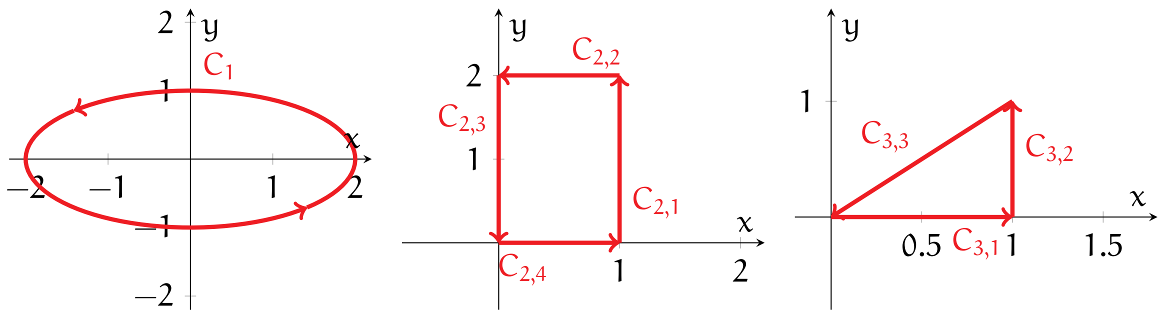
(b) Suppose you apply Stokes' theorem to each of the surfaces S_i in Question (5b), where:

- For S_1, we use the orientation facing the positive x-direction.
- For S_2, we use the outward-facing orientation.
- For S_3, we use the orientation facing the positive z-direction.

Describe the resulting orientation(s) on the boundary of each S_i that you would obtain, according to the statement of Stokes' theorem.

(c) Suppose you apply the divergence theorem to each of the regions V_i in Question (5c). Describe the resulting orientation(s) on the boundary of each V_i that you would obtain, according to the statement of the divergence theorem.

(a) Here, we use the notations from the solution of Question (5a) for our boundaries. First, illustrations of all the orientations are drawn below:



(i) For C_1, we take the anticlockwise orientation.

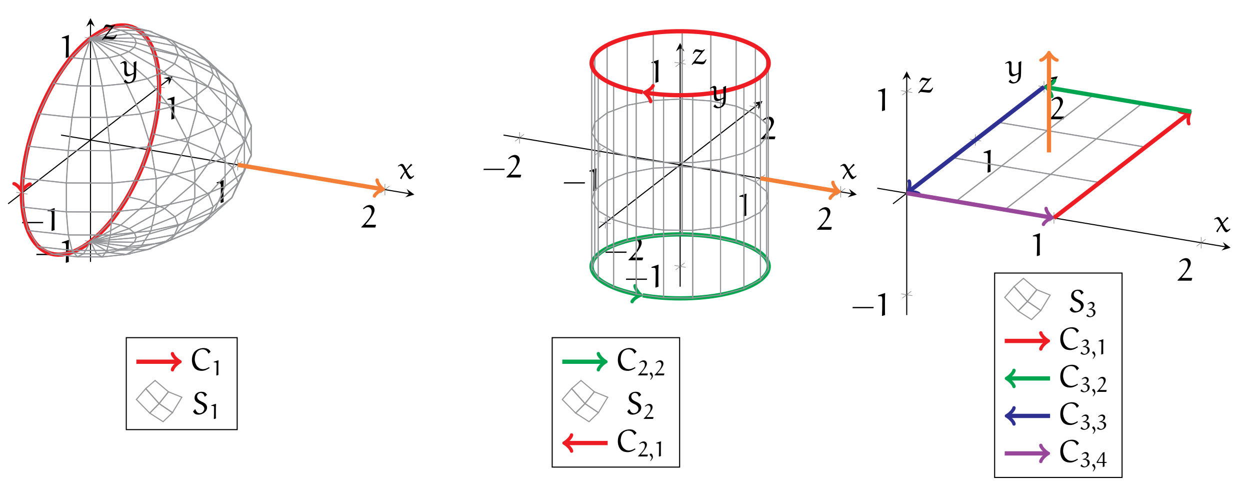
(ii) We traverse around D_2 in the anticlockwise direction. More specifically:

- We take the upward orientation on $C_{2,1}$.
- We take the left orientation on $C_{2,2}$.
- We take the downward orientation on $C_{2,3}$.
- We take the rightward orientation on $C_{2,4}$.

(ii) We traverse around D_3 in the anticlockwise direction. More specifically:

- We take the rightward orientation on $C_{3,1}$.
- We take the upward orientation on $C_{3,2}$.
- We take the downward-leftward orientation on $C_{3,3}$.

(b) Here, we use the notations from the solution of Question (5b) for our boundaries. All the orientations are illustrated below (the orientations of S_1, S_2, S_3 are given as orange arrows):



(i) The boundary curve C_1 has the anticlockwise orientation on the yz-plane, as viewed from someone standing "to the right" (higher value of x) of the half-sphere.

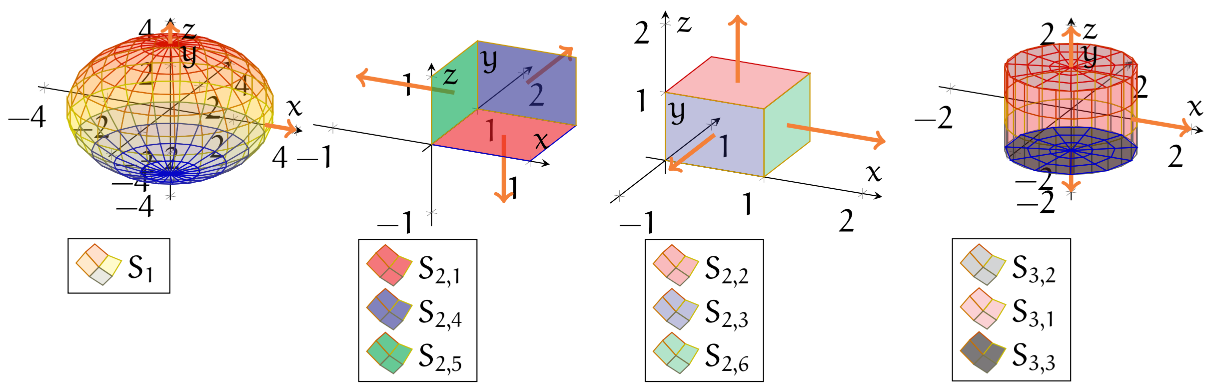
(ii) The orientations of the boundary pieces $C_{2,1}$ and $C_{2,2}$ are assigned separately:

- $C_{2,1}$ has the clockwise orientation with respect to the xy-plane (i.e. one starts from $(x, y) = (1, 0)$ and goes into the fourth quadrant $x > 0$ and $y < 0$).
- $C_{2,2}$ has the anticlockwise orientation with respect to the xy-plane (i.e. one starts from $(x, y) = (1, 0)$ and goes into the first quadrant $x > 0$ and $y > 0$).

(iii) Here, one goes anticlockwise about the square S_3, when one views S_3 "from the top" (i.e. from a higher value of z). More specifically:

- $C_{3,1}$ goes in the positive y-direction.
- $C_{3,2}$ goes in the negative x-direction.
- $C_{3,3}$ goes in the negative y-direction.
- $C_{3,4}$ goes in the positive x-direction.

(c) Here, we use the notations from the solution of Question (5c) for our boundaries. Again, all of the orientations are illustrated below as orange arrows:



(i) For S_1, we simply obtain the outward-facing orientation.

(ii) Each of the boundary pieces is given the outward orientation. More specifically:

- On $S_{2,1}$, we take the unit normals in the negative z-direction.

- On $S_{2,2}$, we take the unit normals in the positive z-direction.
- On $S_{2,3}$, we take the unit normals in the negative y-direction.
- On $S_{2,4}$, we take the unit normals in the positive y-direction.
- On $S_{2,5}$, we take the unit normals in the negative x-direction.
- On $S_{2,6}$, we take the unit normals in the positive x-direction.

(iii) Each of the boundary pieces is given the outward orientation. More specifically:

- On $S_{3,1}$, we take the unit normals in the outward-facing direction.
- On $S_{3,2}$, we take the unit normals in the positive z-direction.
- On $S_{3,3}$, we take the unit normals in the negative z-direction.

(7) *(Second derivative identities)*

(a) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. Show that the curl of the gradient of f vanishes everywhere, that is, show that for any $\mathbf{p} \in \mathbb{R}^3$,

$$[\nabla \times (\nabla f)](\mathbf{p}) = (0, 0, 0)_{\mathbf{p}}.$$

(b) Let $\mathbf{F}$ be a smooth vector field on $\mathbb{R}^3$. Show that the divergence of the curl of $\mathbf{F}$ vanishes everywhere, that is, show that for any $\mathbf{p} \in \mathbb{R}^3$,

$$[\nabla \cdot (\nabla \times \mathbf{F})](\mathbf{p}) = 0.$$

(a) Recall that the gradient of f is given by

$$\nabla f(\mathbf{p}) = (\partial_1 f(\mathbf{p}), \partial_2 f(\mathbf{p}), \partial_3 f(\mathbf{p}))_{\mathbf{p}}.$$

Thus, by the definition of the curl, we obtain

$$\begin{aligned}[\nabla \times (\nabla f)](\mathbf{p}) &= (\partial_2\partial_3 f(\mathbf{p}) - \partial_3\partial_2 f(\mathbf{p}), \partial_3\partial_1 f(\mathbf{p}) - \partial_1\partial_3 f(\mathbf{p}), \partial_1\partial_2 f(\mathbf{p}) - \partial_2\partial_1 f(\mathbf{p}))_{\mathbf{p}} \\ &= (0, 0, 0)_{\mathbf{p}},\end{aligned}$$

since the order that one takes partial derivatives of a function does not matter.

(b) Let us write $\mathbf{F}$ in terms of its components as

$$\mathbf{F}(\mathbf{p}) = (F_1(\mathbf{p}),\, F_2(\mathbf{p}),\, F_3(\mathbf{p}))_{\mathbf{p}}.$$

Recall that the curl of $\mathbf{F}$ is given by

$$(\nabla \times \mathbf{F})(\mathbf{p}) = (\partial_2 F_3(\mathbf{p}) - \partial_3 F_2(\mathbf{p}),\, \partial_3 F_1(\mathbf{p}) - \partial_1 F_3(\mathbf{p}),\, \partial_1 F_2(\mathbf{p}) - \partial_2 F_1(\mathbf{p}))_{\mathbf{p}}.$$

Then, by the definition of the divergence, we have that

$$\begin{aligned}
[\nabla \cdot (\nabla \times \mathbf{F})](\mathbf{p}) &= \partial_1(\partial_2 F_3 - \partial_3 F_2)(\mathbf{p}) + \partial_2(\partial_3 F_1 - \partial_1 F_3)(\mathbf{p}) + \partial_3(\partial_1 F_2 - \partial_2 F_1)(\mathbf{p}) \\
&= \partial_1\partial_2 F_3(\mathbf{p}) - \partial_1\partial_3 F_2(\mathbf{p}) + \partial_2\partial_3 F_1(\mathbf{p}) - \partial_2\partial_1 F_3(\mathbf{p}) \\
&\qquad + \partial_3\partial_1 F_2(\mathbf{p}) - \partial_3\partial_2 F_1(\mathbf{p}) \\
&= 0,
\end{aligned}$$

(Again, the order that one applies partial derivatives does not matter.)

(8) *(Connections to Complex Variables)* *(Not examinable)* Consider the plane $\mathbb{R}^2$, or equivalently, the complex plane $\mathbb{C}$. Let C denote the unit circle centred at the origin,

$$C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \simeq \{z \in \mathbb{C} \mid |z| = 1\},$$

and assign to C the anticlockwise parametrisation. Furthermore, let $f : \mathbb{C} \to \mathbb{C}$ be a complex-analytic function, and write f in terms of its components as

$$f(z) = u(z) + i \cdot v(z), \qquad z \in \mathbb{C}.$$

- **(a)** Express the real and imaginary parts of the contour integral of f over C,

  $$\mathrm{Re} \int_C f(z)\, dz, \qquad \mathrm{Im} \int_C f(z)\, dz,$$

  as curve integrals of appropriate vector fields over C.

- **(b)** Use Green's theorem to prove Cauchy's theorem on C:

  $$\int_C f(z)\, dz = 0.$$

(Hint: Recall that u and v satisfy the Cauchy-Riemann equations.)

(a) To parametrise C for our contour and curve integrals, we can take

$$\gamma : (0, 2\pi) \to \mathbb{R}^2, \qquad \gamma(t) = (x(t), y(t)) = (\cos t, \sin t),$$

or, in terms of complex numbers,

$$\gamma(t) = x(t) + i \cdot y(t) = e^{it}.$$

In particular, we can expand

$$\begin{aligned} \int_C f(z)\, dz &= \int_0^{2\pi} f(\gamma(t))\, \gamma'(t)\, dt \\ &= \int_0^{2\pi} [u(\gamma(t)) + i \cdot v(\gamma(t))][x'(t) + i \cdot y'(t)]\, dt \\ &= \int_0^{2\pi} [u(\gamma(t))\, x'(t) - v(\gamma(t))\, y'(t)]\, dt \\ &\qquad + i \int_0^{2\pi} [u(\gamma(t))\, y'(t) + v(\gamma(t))\, x'(t)]\, dt. \end{aligned}$$

Note that the two integrals on the right-hand side are now real-valued.

Define now the vector fields $\mathbf{R}$ and $\mathbf{I}$ on $\mathbb{R}^2$ by

$$\mathbf{R}(x, y) = (u(x, y), -v(x, y))_{(x,y)}, \qquad \mathbf{I}(x, y) = (v(x, y), u(x, y))_{(x,y)}.$$

Then, combining the above and recalling the definition of curve integrals, we see that

$$\begin{aligned} \operatorname{Re} \int_C f(z)\, dz &= \int_0^{2\pi} [(u(\gamma(t)), -v(\gamma(t))) \cdot \gamma'(t)]\, dt = \int_C \mathbf{R} \cdot d\mathbf{s}, \\ \operatorname{Im} \int_C f(z)\, dz &= \int_0^{2\pi} [(v(\gamma(t)), u(\gamma(t))) \cdot \gamma'(t)]\, dt = \int_C \mathbf{I} \cdot d\mathbf{s}. \end{aligned}$$

(b) Let D denote the interior of C:

$$D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}.$$

Then, recalling the results of part (a) and applying Green's theorem, we see that

$$\mathrm{Re} \int_C f(z)\, dz = \int_C \mathbf{R} \cdot \mathbf{ds} = -\iint_D [\partial_1 v(x,y) + \partial_2 u(x,y)] dx dy,$$
$$\mathrm{Im} \int_C f(z)\, dz = \int_C \mathbf{I} \cdot \mathbf{ds} = \iint_D [\partial_1 u(x,y) - \partial_2 v(x,y)] dx dy.$$

Now, since f is analytic, its components satisfy the Cauchy-Riemann equations:

$$\partial_1 u(x,y) - \partial_2 v(x,y) = 0, \qquad \partial_1 v(x,y) + \partial_2 u(x,y) = 0.$$

Combining all the above, we conclude that

$$\mathrm{Re} \int_C f(z)\, dz = \mathrm{Im} \int_C f(z)\, dz = 0, \qquad \int_C f(z)\, dz = 0.$$

(9) *(Green's theorem fail)* Let C be as in Question (8), and let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)_{(x,y)}, \qquad (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}.$$

(a) Show that for any $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$,

$$\partial_x \left(\frac{x}{x^2+y^2} \right) - \partial_y \left(-\frac{y}{x^2+y^2} \right) = 0.$$

(b) On the other hand, show that the integral of $\mathbf{F}$ over C is *not* zero. Why does this not contradict the statement of Green's theorem? *(More specifically, why do Green's theorem and part (a) not imply that the integral of* $\mathbf{F}$ *over* C *is zero?)*

(c) *(Not examinable)* For those taking *Complex Variables*, can you relate what you saw in parts (a) and (b) to some contour integrals that you have seen?

(a) This is a direct computation using the quotient rule:

$$\begin{aligned} \partial_x \left(\frac{x}{x^2+y^2} \right) - \partial_y \left(-\frac{y}{x^2+y^2} \right) &= \frac{(x^2+y^2) \cdot 1 - 2x \cdot x}{(x^2+y^2)^2} + \frac{(x^2+y^2) \cdot 1 - 2y \cdot y}{(x^2+y^2)^2} \\ &= \frac{-x^2+y^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} \\ &= 0. \end{aligned}$$

(b) Let γ be as in the solution of Question (8a). Then, a direct computation yields

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{s} &= \int_0^{2\pi}\left[\mathbf{F}(\gamma(t))\cdot\gamma'(t)_{\gamma(t)}\right]dt\\ &= \int_0^{2\pi}\left[\left(-\frac{\sin t}{\cos^2 t+\sin^2 t}, \frac{\cos t}{\cos^2 t+\sin^2 t}\right)\cdot(-\sin t, \cos t)\right]dt\\ &= \int_0^{2\pi} 1\, dt.\end{aligned}$$

As a result, we see that

$$\int_C \mathbf{F}\cdot d\mathbf{s} = 2\pi \neq 0.$$

This does not contradict Green's theorem, because the vector field $\mathbf{F}$ fails to be well-defined and smooth in all of the interior of C (in particular, $\mathbf{F}$ does not exist at $(x, y) = (0, 0)$). This is one of the assumptions required for Green's theorem to be applicable.

(c) An equivalent phenomenon in complex variables is the contour integral of the function $f(z) = \frac{1}{z}$ over the unit circle, which is also nonzero. Note that f is clearly analytic, but it fails to be defined at the origin, so Cauchy's theorem does not apply here.