MTH5113 (2023/24): Problem Sheet 10
Solutions

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(1) *(Warm-up)* Consider the following constrained optimisation problem:

- *Maximise the function* $f(x, y) = x$*, subject to the constraint* $x^2 + y^2 = 1$.

(a) Give the solution to the above problem *without doing any calculations. (The answer should be obvious from inspection alone; draw a picture if you are not sure.)*

(b) Solve the above problem using the method of Lagrange multipliers. Verify that your solution matches what you deduced in part (a).

(a) Here, we are simply asked what is the largest possible x-coordinate on the unit circle $x^2 + y^2 = 1$ centred at the origin. Of course, this is achieved at the rightmost point $(1, 0)$, so the maximum value of $f(x, y) = x$ is just $+1$.

(b) First, observe that the constraint curve $x^2 + y^2 = 1$ is a circle, which is both closed and bounded. As a result, we know that the desired maximum of f does indeed exist.

We now apply the method of Lagrange multipliers. Consider the functions

$$\begin{aligned} f : \mathbb{R}^2 \to \mathbb{R}, \qquad & f(x, y) = x, \\ g : \mathbb{R}^2 \to \mathbb{R}, \qquad & g(x, y) = x^2 + y^2. \end{aligned}$$

Note that the gradients of f and g satisfy

$$\nabla f(x, y) = (1,\ 0)_{(x,y)}, \qquad \nabla g(x, y) = (2x,\ 2y)_{(x,y)}.$$

As a result, we must solve the system

$$1 = \lambda \cdot 2x, \qquad 0 = \lambda \cdot 2y, \qquad x^2 + y^2 = 1,$$

for the unknowns $x, y, \lambda \in \mathbb{R}$.

First, note that the equation $1 = 2\lambda x$ implies that $\lambda \neq 0$. Combining this with the second equation $2\lambda y = 0$ then yields $y = 0$. Putting this into the constraint equation, we conclude that $x^2 = 1$, and hence $x = \pm 1$. Since we have x, we can also now use the first equation $1 = \lambda \cdot 2x$ to solve for λ. From this, we conclude that the solutions to the system are

$$(x, y, \lambda) = \left(+1,\ 0,\ +\frac{1}{2} \right), \qquad (x, y, \lambda) = \left(-1,\ 0,\ -\frac{1}{2} \right).$$

In particular, the maximum of x could only be achieved at the points $(x, y) = (\pm 1, 0)$.

We can now check both points and compare:

$$f(+1, 0) = +1, \qquad f(-1, 0) = -1.$$

Thus, we see that the maximum of value of f, subject to the constraint $x^2 + y^2 = 1$, is $+1$, and that this maximum is achieved at $(x, y) = (+1, 0)$.

(2) *(Warm-up)* Consider the following constrained optimisation problem:

- *Minimise the function* $f(x, y, z) = z$*, subject to the constraint* $x^2 + y^2 + z^2 = 1$.

(a) Give the solution to the above problem *without doing any calculations. (The answer should be obvious from inspection alone; draw a picture if you are not sure.)*

(b) Solve the above problem using the method of Lagrange multipliers. Verify that your solution matches what you deduced in part (a).

(a) Here, the goal is to find the smallest z-value on the unit sphere $x^2 + y^2 + z^2 = 1$ centred at the origin. This is achieved at the south pole $(0, 0, -1)$, with minimum value -1.

(b) First, the constraint surface $x^2 + y^2 + z^2 = 1$ is a sphere, which is both closed and bounded. As a result, our desired minimum of f is guaranteed to exist.

To apply the method of Lagrange multipliers, we consider the functions

$$\begin{aligned} f &: \mathbb{R}^3 \to \mathbb{R}, \qquad f(x, y, z) = z, \\ g &: \mathbb{R}^3 \to \mathbb{R}, \qquad g(x, y, z) = x^2 + y^2 + z^2. \end{aligned}$$

Note that the gradients of f and g satisfy

$$\nabla f(x, y) = (0,\ 0,\ 1)_{(x,y,z)}, \qquad \nabla g(x, y) = (2x,\ 2y,\ 2z)_{(x,y,z)}.$$

Thus, our goal is to solve the following system for the unknowns $x, y, z, \lambda \in \mathbb{R}$:

$$0 = \lambda \cdot 2x, \qquad 0 = \lambda \cdot 2y, \qquad 1 = \lambda \cdot 2z, \qquad x^2 + y^2 + z^2 = 1.$$

First, the equation $1 = 2\lambda z$ implies $\lambda \neq 0$. Combining this with the first two equations yields $x = y = 0$. Putting this into the constraint equation, we see that $z^2 = 1$, and thus $z = \pm 1$. From all the above, we conclude that the solutions to the above system are

$$(x, y, z, \lambda) = \left(0,\ 0,\ +1,\ +\frac{1}{2}\right), \qquad (x, y, z, \lambda) = \left(0,\ 0,\ -1,\ -\frac{1}{2}\right).$$

Finally, we check the vales of f at the above-mentioned points:

$$f(0, 0, +1) = +1, \qquad f(0, 0, -1) = -1.$$

Thus, we conclude that f is minimised at $(0, 0, -1)$, with minimum value -1.

(3) [**Marked**] Solve the following problem using the method of Lagrange multipliers:

- *Find the maximum and minimum values of* x^2y^2*, subject to the constraint*

$$\left(\left(x - \frac{3}{2} \right)^2 + y^2 \right) \left(\left(x + \frac{3}{2} \right)^2 + y^2 \right) = 9 \,.$$

At which points are the maximum and minimum values achieved?

First, note that the constraint curve

$$C = \left\{ (x, y) \in \mathbb{R}^2 \mid \left(\left(x - \frac{3}{2} \right)^2 + y^2 \right) \left(\left(x + \frac{3}{2} \right)^2 + y^2 \right) = 9 \right\}$$

traces out a peanut, and hence is closed and bounded. As a result, the maximum and minimum values of x^2y^2 are guaranteed to exist on C. [1 mark for this observation]

We now apply the method of Lagrange multipliers. First, we define

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad f(x, y) = x^2y^2,$$
$$g : \mathbb{R}^2 \to \mathbb{R}, \qquad g(x, y) = \left(\left(x - \frac{3}{2} \right)^2 + y^2 \right) \left(\left(x + \frac{3}{2} \right)^2 + y^2 \right).$$

Note that the gradients of f and g satisfy

$$\nabla f(x,y) = (2xy^2,\, 2x^2y)_{(x,y)}, \qquad \nabla g(x,y) = \left(x\left(4x^2+4y^2-9\right), y\left(4x^2+4y^2+9\right)\right)_{(x,y)}.$$

Thus, our goal is to solve the following system for $x, y, \lambda \in \mathbb{R}$:

$$2xy^2 = \lambda \cdot x\left(4x^2+4y^2-9\right), \qquad 2x^2y = \lambda \cdot y\left(4x^2+4y^2+9\right),$$
$$\left(\left(x-\frac{3}{2}\right)^2 + y^2\right)\left(\left(x+\frac{3}{2}\right)^2 + y^2\right) = 9.$$

[1 mark for correct system of equations]

As we will be dividing by both x and y, we split into cases:

- If $x = 0$, then the third equation yields $\left(y^2 + \frac{9}{4}\right)^2 = 9$, which means $y = \pm\frac{\sqrt{3}}{2}$. Putting this into the second equation gives $\lambda = 0$.

  As a result, we obtain two solutions when $x = 0$:

  $$(x,y,\lambda) = \left(0, \frac{\pm\sqrt{3}}{2}, 0\right).$$

- If $y = 0$, then the third equation yields $\left(x^2 - \frac{9}{4}\right)^2 = 9$, or $x = \pm\frac{\sqrt{21}}{2}$. Putting this into the first equation gives $\lambda = 0$.

  As a result, we obtain two solutions when $y = 0$:

  $$(x,y,\lambda) = \left(\pm\frac{\sqrt{21}}{2}, 0, 0\right).$$

- If $x \neq 0$ and $y \neq 0$, then we can divide the first and second equations by x and y, respectively. Then, the first and second equations combined yield the relations

  $$2y^2 = \lambda \cdot \left(4x^2+4y^2-9\right), \qquad 2x^2 = \lambda \cdot \left(4x^2+4y^2+9\right)$$

  Which we can rewrite as:

  $$4\lambda x^2 - (2-4\lambda)y^2 = 9\lambda, \qquad (2-4\lambda)x^2 - 4\lambda y^2 = 9\lambda\ .$$

  We immediately notice that these are not independent equations if $\lambda = \frac{1}{4}$, so for now we will assume $\lambda \neq 1/4$. We will also assume $\lambda \neq 0$ [1 mark for noticing this subtlety.]

Solving the first equation for x^2 gives $x^2 = \frac{9}{4} + \left(\frac{1}{2\lambda} - 1\right)y^2$.

Plugging this into the second equation then yields

$$\frac{9}{2} + y^2\left(\frac{1}{\lambda} - 4\right) = 18\lambda, \qquad \implies y^2 = -\frac{9\lambda}{2}.$$

If we plug this back into the equation for x^2 we find:

$$x^2 = +\frac{9\lambda}{2}$$

So, if $\lambda \in \mathbb{R}$, one of x or y must be pure imaginary. Therefore we have no real solution for $\lambda \neq 1/4$.

Our last hope is to check the case $\lambda = \frac{1}{4}$.

When $\lambda = \frac{1}{4}$, the first and second equations are the same and can be solved to give :

$$x^2 = \frac{9}{4} + y^2 \ .$$

Plugging this into the third equation gives:

$$y^2\left(9 + 4y^2\right) = 9$$

Using the quadratic formula we find the real solutions are $y = \pm\frac{\sqrt{3}}{2}$, and plugging into the previous formula, we find $x = \pm\sqrt{3}$. We thus have four solutions of this type.

[2 mark for solving system mostly correctly]

It remains to compute the values of f at all these solution points:

$$f\left(0, \pm\frac{\sqrt{3}}{2}\right) = 0, \qquad f\left(\pm\frac{\sqrt{21}}{2}, 0\right) = 0,$$
$$f\left(+\sqrt{3}, +\frac{\sqrt{3}}{2}\right) = +\frac{9}{4}, \qquad f\left(-\sqrt{3}, +\frac{\sqrt{3}}{2}\right) = +\frac{9}{4},$$
$$f\left(+\sqrt{3}, -\frac{\sqrt{3}}{2}\right) = +\frac{9}{4}, \qquad f\left(-\sqrt{3}, -\frac{\sqrt{3}}{2}\right) = +\frac{9}{4}.$$

Since the maximum and minimum values are guaranteed to exist, we obtain the following solutions to our constrained optimisation problem:

- The maximum value is $+\frac{9}{4}$, and this is achieved at $\left(+\sqrt{3}, +\frac{\sqrt{3}}{2}\right)$, $\left(+\sqrt{3}, -\frac{\sqrt{3}}{2}\right)$, $\left(0, \pm\frac{\sqrt{3}}{2}\right)$ and $\left(-\sqrt{3}, -\frac{\sqrt{3}}{2}\right)$.
- The minimum value is 0, and this is achieved at $\left(0, \pm\frac{\sqrt{3}}{2}\right)$ and $\left(\pm\frac{\sqrt{21}}{2}, 0\right)$.

[1 mark for mostly correct answer]

(4) *(Differential Geometry and Game Theory)* Let x^2 be the number of hours of *MTH5113* lectures and tutorials you attend, and let y^2 be the number of hours of *MTH5113* lectures and tutorials you skip. As you know, the *constraint* is that there are only 43 total hours of lectures and tutorials in *MTH5113*. Now, suppose that the "effectiveness" of your learning in *MTH5113*, as a function of the above hours spent, is modelled by the relation

$$E = 100x^2 + y^2.$$

(Here, a higher value of E *means better learning!)* Your objective here is to *optimise* the "effectiveness" of your learning experience in *MTH5113*!

(a) Express the above objective as a constrained optimisation problem.

(b) Use the method of Lagrange multipliers to solve the problem in part (a).

(c) Given your answer in (b), what optimal strategy should you adopt in order to have the most effective learning experience in *MTH5113*? :)

(a) Consider the functions

$$E : \mathbb{R}^2 \to \mathbb{R}, \qquad E(x, y) = 100x^2 + y^2,$$
$$g : \mathbb{R}^2 \to \mathbb{R}, \qquad g(x, y) = x^2 + y^2.$$

The problem, then, is to maximise $E(x, y)$, subject to the constraint $g(x, y) = 43$.

(b) First, since the constraint curve $x^2 + y^2 = 43$ is a circle, which is closed and bounded, we know that a maximum value of E is indeed achieved. Thus, we apply the method of Lagrange multipliers to find this maximum. For this, we take the gradients of E and g, which indicates that we must following solve the system of equations:

$$200x = 2\lambda x, \qquad 2y = 2\lambda y, \qquad x^2 + y^2 = 43.$$

Suppose first that both x and y are nonzero. Then, the first two equations imply

$$100 = \frac{200x}{2x} = \lambda = \frac{2y}{2y} = 1,$$

which is a contradiction. Thus, either $x = 0$ or $y = 0$. Now:

- If $x = 0$, then the constraint equation yields $y = \pm\sqrt{43}$. (Note that the first two equations in the system imply $\lambda = 1$ but do not further restrict y.)
- If $y = 0$, then the constraint equation yields $x = \pm\sqrt{43}$. (Note that the first two equations in the system imply $\lambda = 100$ but do not further restrict x.)

Thus, the solutions to our system are given by

$$(x, y, \lambda) = \left(\pm\sqrt{43},\, 0,\, 100\right), \qquad (x, y) = \left(0,\, \pm\sqrt{43},\, 1\right).$$

It remains to check which solution maximises "effectiveness":

$$E\left(\pm\sqrt{43},\, 0\right) = 4300, \qquad E\left(0,\, \pm\sqrt{43}\right) = 43.$$

Thus, the maximum "effectiveness" is 4300, which is achieved at $(x^2, y^2) = (43, 0)$.

(c) The results in part (b) show that the maximum "effectiveness" is achieved when $x^2 = 43$ hours are spent in lectures and tutorials, and $y^2 = 0$ hours are spent skipping lectures and tutorials. Thus, the optimal strategy is to attend all the lectures and tutorials!

(5) [**Tutorial**] Use the method of Lagrange multipliers to solve the following:

(a) Find the minimum and maximum of $4x^2 - y^2$, subject to the constraint $x^2 + 4y^2 = 4$.

(b) Find the unit vectors $(x, y, z) \in \mathbb{R}^3$ that maximise and minimise the dot product,

$$(6, -3, 2) \cdot (x, y, z).$$

(a) First, the constraint curve $x^2 + 4y^2 = 4$ is an ellipse, which is closed and bounded, hence both the maximum and minimum in our problem exist.

To apply the method of Lagrange multipliers, we consider the functions

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad f(x, y) = 4x^2 - y^2,$$
$$g : \mathbb{R}^2 \to \mathbb{R}, \qquad g(x, y) = x^2 + 4y^2.$$

Taking gradients of f and g, we see that we must solve the system

$$8x = 2\lambda x, \qquad -2y = 8\lambda y, \qquad x^2 + 4y^2 = 4.$$

We break this analysis into cases:

- If $x = 0$, then the constraint implies $y = \pm 1$; this yields $(x, y, \lambda) = (0, \pm 1, -\frac{1}{4})$.
- If $y = 0$, then the constraint implies $x = \pm 2$; this yields $(x, y, \lambda) = (\pm 2, 0, 4)$.
- If $x \neq 0$ and $y \neq 0$, then the first two equations yield

  $$4 = \frac{8x}{2x} = \lambda = \frac{-2y}{8y} = -\frac{1}{4},$$

  which is a contradiction.

As a result, the solutions of the above system are

$$(x, y, \lambda) = \left(0, \pm 1, -\frac{1}{4}\right), \qquad (x, y) = (\pm 2, 0, 4).$$

Finally, plugging each of the above into f, we see that

$$f(0, \pm 1) = -1, \qquad f(\pm 2, 0) = 16.$$

Thus, the maximum value is 16, and the minimum value is -1.

(b) First, we reformulate the question as a constrained optimisation problem. In particular, our objective is to maximise and minimise

$$f(x, y, z) = (6, -3, 2) \cdot (x, y, z) = 6x - 3y + 2z,$$

subject to the constraint (i.e. that (x, y, z) is a unit vector)

$$g(x, y, z) = x^2 + y^2 + z^2 = 1.$$

In particular, the constraint surface is the unit sphere, which is closed and bounded, hence our desired maximum and minimum values are guaranteed to exist.

By the method of Lagrange multipliers, we must solve the system

$$6 = 2\lambda x, \qquad -3 = 2\lambda y, \qquad 2 = 2\lambda z, \qquad x^2 + y^2 + z^2 = 1.$$

For this, we first deduce the following:

- If $x = 0$, then the first equation implies $6 = 0$, a contradiction.
- If $y = 0$, then the second equation implies $-3 = 0$, a contradiction.
- If $z = 0$, then the third equation implies $2 = 0$, a contradiction.

Thus, we conclude that $x \neq 0$, $y \neq 0$, and $z \neq 0$. The first three equations then imply

$$\lambda = \frac{3}{x} = \frac{-3}{2y} = \frac{1}{z} \qquad \Rightarrow \qquad x = 3z, \quad y = -\frac{3}{2} \cdot z.$$

Plugging this into the constraint equation yields

$$1 = (3z)^2 + \left(-\frac{3}{2} \cdot z\right)^2 + z^2 = \frac{49}{4} \cdot z^2 \quad \Rightarrow \quad z = \pm\frac{2}{7}.$$

Since $x = 3z$, $y = -\frac{3}{2}z$, and $\lambda = \frac{1}{z}$, we see that the solutions to the system are

$$(x, y, z, \lambda) = \left(+\frac{6}{7}, -\frac{3}{7}, +\frac{2}{7}, +\frac{7}{2}\right), \qquad (x, y, z, \lambda) = \left(-\frac{6}{7}, +\frac{3}{7}, -\frac{2}{7}, -\frac{7}{2}\right).$$

Finally, we check the dot products:

$$\begin{aligned} f\left(+\frac{6}{7}, -\frac{3}{7}, +\frac{2}{7}\right) &= (6, -3, 2) \cdot \left(+\frac{6}{7}, -\frac{3}{7}, +\frac{2}{7}\right) = +7, \\ f\left(-\frac{6}{7}, +\frac{3}{7}, -\frac{2}{7}\right) &= (6, -3, 2) \cdot \left(-\frac{6}{7}, +\frac{3}{7}, -\frac{2}{7}\right) = -7. \end{aligned}$$

Thus, the given dot product is maximised by the unit vector

$$(x_{\max}, y_{\max}, z_{\max}) = \left(+\frac{6}{7}, -\frac{3}{7}, +\frac{2}{7}\right),$$

while this dot product is minimised by the unit vector

$$(x_{\min}, y_{\min}, z_{\min}) = \left(-\frac{6}{7}, +\frac{3}{7}, -\frac{2}{7}\right).$$

(6) *(Conservative and liberal vector fields)*

(a) Let f be the real-valued function

$$f : \mathbb{R}^3 \to \mathbb{R}, \qquad f(x, y, z) = x^4 y^2 z.$$

Compute the integral of the vector field ∇f over the curve

$$C = \{(t, t^2, t^3) \in \mathbb{R}^2 \mid t \in (0, 1)\},$$

where C is given the *rightward* (i.e. in the direction of increasing x-value) *orientation.*

(b) Let $g : \mathbb{R}^2 \to \mathbb{R}$ be given by the formula

$$g(x, y) = x^{17} e^{y + x^2 y^5 \cos x^7} + y^4 + e^{x^2 + y^2 + x^{42} + y^{1776} e^{yx^2}}.$$

Find the integral of the vector field ∇g over the unit circle

$$\mathcal{C} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\},$$

where $\mathcal{C}$ is given the *anticlockwise orientation.*

(c) Integrate the vector field

$$\mathbf{H}(x, y) = (-y, x), \qquad (x, y) \in \mathbb{R}^2,$$

over the unit circle $\mathcal{C}$ from part (b), where $\mathcal{C}$ again has the anticlockwise orientation.

(d) From your answer in part (c), conclude that the vector field $\mathbf{H}$ cannot be the gradient of any real-valued function $h : \mathbb{R}^2 \to \mathbb{R}$.

(a) Note C is a bounded, oriented curve, with initial point $(0,0,0)$ and final point $(1,1,1)$. Thus, by the fundamental theorem of calculus (generalised to curve integrals), we have

$$\int_C \nabla f \cdot \mathbf{ds} = f(1,1,1) - f(0,0,0) = 1 - 0 = 1.$$

(b) Note that we can remove a single point from $\mathcal{C}$ without altering the values of any integrals over $\mathcal{C}$. With this point removed, the circle then becomes a bounded, oriented curve with a common initial and final point $\mathbf{p}$. As a result,

$$\int_C \nabla g \cdot \mathbf{ds} = g(\mathbf{p}) - g(\mathbf{p}) = 0.$$

(c) The only thing that can be done is to compute this integral directly. Note that

$$\gamma : (0, 2\pi) \to \mathcal{C}, \qquad \gamma(t) = (\cos t, \sin t)$$

is an injective parametrisation of $\mathcal{C}$ whose image is "almost all" of $\mathcal{C}$, and that γ generates our given anticlockwise orientation of $\mathcal{C}$. Furthermore, note that

$$\mathbf{H}(\gamma(t)) \cdot \gamma'(t)_{\gamma(t)} = (-\sin t, \cos t) \cdot (-\sin t, \cos t) = 1.$$

As a result, we conclude that

$$\int_{\mathcal{C}} \mathbf{H} \cdot \mathbf{ds} = + \int_0^{2\pi} [\mathbf{H}(\gamma(t)) \cdot \gamma'(t)_{\gamma(t)}] dt = \int_0^{2\pi} dt = 2\pi.$$

(d) If $\mathbf{H} = \nabla h$ for some $h : \mathbb{R}^2 \to \mathbb{R}$, then the same argument as in part (b), using the fundamental theorem of calculus, would imply that

$$\int_{\mathcal{C}} \mathbf{H} \cdot \mathbf{ds} = 0,$$

which would contradict our result in (c). Thus, $\mathbf{H}$ is not a gradient.

(7) *(Lagrangian Formulation of Multipliers)* Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be smooth functions, and fix $c \in \mathbb{R}$. Show that the following conditions are equivalent:

(i) $(x,y) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$ satisfy the following system of equations:

$$\nabla f(x,y) = \lambda \cdot \nabla g(x,y), \qquad g(x,y) = c.$$

(ii) $(x,y,\lambda) \in \mathbb{R}^3$ satisfies the equation

$$\nabla \mathcal{L}(x,y,\lambda) = (0,\, 0,\, 0)_{(x,y,\lambda)},$$

where the function $\mathcal{L}$, called the *Lagrangian*, is defined by

$$\mathcal{L} : \mathbb{R}^3 \to \mathbb{R}, \qquad \mathcal{L}(u,v,w) = f(u,v) - w[g(u,v) - c].$$

(Thus, the method of Lagrange multipliers could also be formulated in terms of $\mathcal{L}$ in (ii)—if the maximum or minimum of f, subject to the constraint g, is achieved at (x,y), then there is some $\lambda \in \mathbb{R}$ such that (x,y,λ) is a critical point of the Lagrangian $\mathcal{L}$.)

We begin by differentiating $\mathcal{L}$:

$$\begin{aligned}
\partial_1 \mathcal{L}(u,v,w) &= \partial_u \{f(u,v) - w[g(u,v) - c]\} = \partial_1 f(u,v) - w \cdot \partial_1 g(u,v), \\
\partial_2 \mathcal{L}(u,v,w) &= \partial_v \{f(u,v) - w[g(u,v) - c]\} = \partial_2 f(u,v) - w \cdot \partial_2 g(u,v), \\
\partial_3 \mathcal{L}(u,v,w) &= \partial_w \{f(u,v) - w[g(u,v) - c]\} = -g(u,v) + c.
\end{aligned}$$

As a result,

$$\nabla \mathcal{L}(u,v,w) = (\partial_1 f(u,v) - w\partial_1 g(u,v),\, \partial_2 f(u,v) - w\partial_2 g(u,v),\, -g(u,v) + c)_{(u,v,w)}.$$

Consequently, $\nabla \mathcal{L}(x,y,\lambda)$ vanishes if and only if

$$\partial_1 f(x,y) - \lambda \cdot \partial_1 g(x,y) = 0, \qquad \partial_2 f(x,y) - \lambda \cdot \partial_2 g(x,y) = 0, \qquad - g(x,y) + c = 0.$$

Similarly, note that (x,y) and λ satisfy the system in (i) if and only if

$$\partial_1 f(x,y) = \lambda \cdot \partial_1 g(x,y), \qquad \partial_2 f(x,y) = \lambda \cdot \partial_2 g(x,y), \qquad g(x,y) = c.$$

(In particular, we used the definitions of the gradients of f and g.) Clearly, the two systems of equations derived above are the same, hence it follows that (i) and (ii) are equivalent.

(8) *(Multiple Constraints)* Assume the following formal setting:

- Let $U \subseteq \mathbb{R}^3$ be open and connected.
- Let $f : U \to \mathbb{R}$, $g : U \to \mathbb{R}$, $h : U \to \mathbb{R}$ be smooth functions.
- Suppose $\nabla g(\mathbf{p}) \times \nabla h(\mathbf{p})$ is nonvanishing at every $\mathbf{p} \in U$.

Under the above assumptions, the following result holds:

- **Theorem.** Suppose f achieves its maximum or minimum value on

$$C = \{(x, y, z) \in U \mid g(x, y, z) = c,\ h(x, y, z) = d\}$$

at a point $\mathbf{p} \in C$. Then, there exist $\lambda, \mu \in \mathbb{R}$ such that

$$\nabla f(\mathbf{p}) = \lambda \cdot \nabla g(\mathbf{p}) + \mu \cdot \nabla h(\mathbf{p}).$$

Using the preceding theorem:

(a) Devise a corresponding *method of Lagrange multipliers* for solving the following constraint optimisation problem: *maximise or minimise* $f(x, y, z)$*, subject to the simultaneous constraints* $g(x, y, z) = c$ *and* $h(x, y, z) = d$.

(b) *Using your strategy from part (a)*, find the maximum and minimum values of $x+y+z$, subject to the simultaneous constraints $x^2 + y^2 = 1$ and $x - z = 1$.

(a) Here, the main idea is that we only need to check the points $\mathbf{p} \in C$ satisfying

$$\nabla f(\mathbf{p}) = \lambda \cdot \nabla g(\mathbf{p}) + \mu \cdot \nabla h(\mathbf{p}).$$

Thus, the corresponding method of Lagrange multipliers is as follows:

- *Step 1:* Solve the following system of equations,

$$\begin{aligned} \partial_1 f(x, y, z) &= \lambda \cdot \partial_1 g(x, y, z) + \mu \cdot \partial_1 h(x, y, z), \\ \partial_2 f(x, y, z) &= \lambda \cdot \partial_2 g(x, y, z) + \mu \cdot \partial_2 h(x, y, z), \\ \partial_3 f(x, y, z) &= \lambda \cdot \partial_3 g(x, y, z) + \mu \cdot \partial_3 h(x, y, z), \\ g(x, y, z) &= c, \\ h(x, y, z) &= d, \end{aligned}$$

for the unknowns $(x, y, z, \lambda, \mu) \in \mathbb{R}^5$.

- *Step 2:* Compute $f(x,y,z)$ for each solution (x,y,z,λ,μ) obtained from step 1. Check which (if any) of the $f(x,y,z)$'s is a maximum and/or a minimum.

(b) First, note that the constraint curve

$$C = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1,\ x - z = 1\}$$

is a slanted cross-section of the unit cylinder about the z-axis; this has the shape of an ellipse, which is closed and bounded. Consequently, the desired maximum and minimum values of $x + y + z$ on this curve C are guaranteed to exist.

In terms of the notations in the question statement, we have

$$f(x,y,z) = x + y + z, \qquad g(x,y,z) = x^2 + y^2, \qquad h(x,y,z) = x - z.$$

Taking partial derivatives, we obtain

$$\begin{aligned}
\partial_1 f(x,y,z) &= 1, & \partial_1 g(x,y,z) &= 2x, & \partial_1 h(x,y,z) &= 1, \\
\partial_2 f(x,y,z) &= 1, & \partial_2 g(x,y,z) &= 2y, & \partial_2 h(x,y,z) &= 0, \\
\partial_3 f(x,y,z) &= 1, & \partial_3 g(x,y,z) &= 0, & \partial_3 h(x,y,z) &= -1,
\end{aligned}$$

Thus, the system of equations we must consider is:

$$1 = 2\lambda x + \mu, \qquad 1 = 2\lambda y, \qquad 1 = -\mu, \qquad x^2 + y^2 = 1, \qquad x - z = 1.$$

To solve the system, we proceed as follows:

- Applying the third equation to the first yields

$$2 = 2\lambda x \quad \Rightarrow \quad 1 = \lambda x.$$

- The above and the second equation imply $x \neq 0$ and $y \neq 0$, hence

$$\frac{1}{2y} = \lambda = \frac{1}{x} \quad \Rightarrow \quad x = 2y.$$

- Putting this into the first constraint yields

$$(2y)^2 + y^2 = 1 \quad \Rightarrow \quad y = \pm\frac{1}{\sqrt{5}}.$$

Since $x = 2y$ and $z = x - 1$, the solutions of the system are thus given by

$$(x, y, z, \lambda, \mu) = \left(+\frac{2}{\sqrt{5}}, +\frac{1}{\sqrt{5}}, +\frac{2}{\sqrt{5}} - 1, +\frac{\sqrt{5}}{2}, -1\right),$$
$$(x, y, z, \lambda, \mu) = \left(-\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} - 1, -\frac{\sqrt{5}}{2}, -1\right).$$

Finally, we check that

$$f\left(+\frac{2}{\sqrt{5}}, +\frac{1}{\sqrt{5}}, +\frac{2}{\sqrt{5}} - 1\right) = +\sqrt{5} - 1,$$
$$f\left(-\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} - 1\right) = -\sqrt{5} - 1.$$

Consequently, subject to the constraints $x^2 + y^2 = 1$ and $x - z = 1$, the maximum value of $x + y + z$ is $+\sqrt{5} - 1$, while the minimum value of $x + y + z$ is $-\sqrt{5} - 1$.

(>9000) *(Extra Exploration)* Put your geometry, calculus, and linear algebra knowledge to the test! Can you prove the theorem stated in Question (8)?

(Hint: The starting point is to observe that C *is a curve, by the result of Question (9) of Problem Sheet 4. From here, you have all the background you need to do this!)*

(Note: While I will not be posting the solution to this problem, I would be happy to chat with anyone who wishes to attempt it. Good luck!)