MTH5113 (2023/24): Problem Sheet 9
Solutions

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(1) *(Warm-up)*

(a) Consider the (real-valued) function

$$F : \mathbb{R}^3 \to \mathbb{R}, \qquad F(x, y, z) = xy^2z^3,$$

as well as the parametric surface

$$\mathbf{P} : (0, 1) \times (0, 1) \to \mathbb{R}^3, \qquad \mathbf{P}(u, v) = (1,\ u,\ v).$$

Compute the surface integral of F over $\mathbf{P}$.

(b) Consider the (real-valued) function

$$G : \mathbb{R}^3 \to \mathbb{R}, \qquad G(x, y, z) = x^2 + y^2,$$

as well as the parametric surface

$$\tau : (0, 2\pi) \times (0, 1) \to \mathbb{R}^3, \qquad \tau(u, v) = (v \cos u,\ v \sin u,\ v).$$

Compute the surface integral of G over τ.

(a) We begin by computing some required quantities. Differentiating $\mathbf{P}$ yields

$$\partial_1 \mathbf{P}(u, v) = (0,\ 1,\ 0), \qquad \partial_2 \mathbf{P}(u, v) = (0,\ 0,\ 1),$$

and their cross product satisfies

$$\partial_1 \mathbf{P}(u, v) \times \partial_2 \mathbf{P}(u, v) = (1,\ 0,\ 0), \qquad |\partial_1 \mathbf{P}(u, v) \times \partial_2 \mathbf{P}(u, v)| = 1.$$

Furthermore, observe that

$$F(\mathbf{P}(u, v)) = F(1,\ u,\ v) = u^2v^3.$$

Thus, by the definition of the (parametric) surface integral, we obtain

$$\begin{aligned}
\int\int_{\mathbf{P}} F\, dA &= \int\int_{(0,1)\times(0,1)} F(\mathbf{P}(u,v))|\partial_1\mathbf{P}(u,v) \times \partial_2\mathbf{P}(u,v)|\, du dv \\
&= \int_0^1\int_0^1 (u^2v^3 \cdot 1)\, du dv \\
&= \int_0^1 u^2\, du \int_0^1 v^3\, dv \\
&= \frac{1}{12}.
\end{aligned}$$

(b) First, we differentiate τ,

$$\partial_1\tau(u,v) = (-v\sin u,\ v\cos u,\ 0), \qquad \partial_2\tau(u,v) = (\cos u,\ \sin u,\ 1),$$

and we compute their cross product:

$$\partial_1\tau(u,v) \times \partial_2\tau(u,v) = (v\cos u,\ v\sin u,\ -v), \qquad |\partial_1\tau(u,v) \times \partial_2\tau(u,v)| = \sqrt{2}\cdot v.$$

In addition, note that

$$G(\tau(u,v)) = G(v\cos u,\ v\sin u,\ v) = v^2\cos^2 u + v^2\sin^2 u = v^2.$$

Using the above, we can now evaluate the given surface integral:

$$\begin{aligned}
\int\int_{\tau} G\, dA &= \int\int_{(0,2\pi)\times(0,1)} (v^2 \cdot \sqrt{2}v)\, du dv \\
&= \sqrt{2}\int_0^{2\pi} du \int_0^1 v^3\, dv \\
&= \sqrt{2}\cdot 2\pi \cdot \frac{1}{4} \\
&= \frac{\pi}{\sqrt{2}}.
\end{aligned}$$

(2) *(Intro to surface integrals)* One can also define an intermediate notion of surface integration of vector fields over *parametric surfaces*. More specifically:

Definition. Let $\sigma : U \to \mathbb{R}^3$ be a parametric surface, and let $\mathbf{F}$ be a vector field that is defined on the image of γ. We then define the *surface integral* of $\mathbf{F}$ over σ by

$$\int\int_{\sigma} \mathbf{F} \cdot d\mathbf{A} = \int\int_{U} \{\mathbf{F}(\sigma(u,v)) \cdot [\partial_1\sigma(u,v) \times \partial_2\sigma(u,v)]_{\sigma(u,v)}\} \, dudv.$$

(a) Consider the vector field $\mathbf{F}$ on $\mathbb{R}^3$ given by

$$\mathbf{F}(x,y,z) = \left(y,\ z^{5800} e^{x^{2000}+46y^{1523}},\ x\right)_{(x,y,z)},$$

and let $\mathbf{P}$ be the parametric plane

$$\mathbf{P} : (0,1) \times (0,1) \to \mathbb{R}^3, \qquad \mathbf{P}(u,v) = (1,\ u,\ v).$$

Compute the surface integral of $\mathbf{F}$ over $\mathbf{P}$.

(b) Consider the vector field $\mathbf{G}$ on $\mathbb{R}^3$ given by

$$\mathbf{G}(x,y,z) = (z,\ z,\ x^2 + y^2)_{(x,y,z)},$$

and let τ be the parametric torus

$$\tau : (0,2\pi) \times (0,1) \to \mathbb{R}^3, \qquad \tau(u,v) = (v\cos u,\ v\sin u,\ v).$$

Compute the surface integral of $\mathbf{G}$ over τ.

(c) Consider the vector field $\mathbf{H}$ on $\mathbb{R}^3$ given by

$$\mathbf{H}(x,y,z) = (-x,\ -y,\ z)_{(x,y,z)},$$

and let $\mathbf{q}$ be the (regular) parametric surface

$$\mathbf{q} : (0,1) \times (0,1) \to \mathbb{R}^3, \qquad \mathbf{q}(u,v) = (u,\ v,\ u^2 + v^2).$$

Compute the surface integral of $\mathbf{H}$ over $\mathbf{q}$.

(a) We begin by computing some preliminary quantities:

$$\begin{aligned} \partial_1\mathbf{P}(u,v) \times \partial_2\mathbf{P}(u,v) &= (0,\ 1,\ 0) \times (0,\ 0,\ 1) \\ &= (1,\ 0,\ 0), \end{aligned}$$

$$
\begin{aligned}
\mathbf{F}(\mathbf{P}(u,v)) &= \mathbf{F}(1,u,v) \\
&= \left(u,\ v^{5800} e^{1+46u^{1523}},\ 1\right)_{\mathbf{P}(u,v)},
\end{aligned}
$$

for any $(u,v) \in (0,1) \times (u,1)$. The above then implies

$$
\begin{aligned}
\mathbf{F}(\mathbf{P}(u,v)) \cdot [\partial_1 \mathbf{P}(u,v) \times \partial_2 \mathbf{P}(u,v)]_{\mathbf{P}(u,v)} &= \left(u,\ v^{5800} e^{1+46u^{1523}},\ 1\right) \cdot (1,\ 0,\ 0) \\
&= u.
\end{aligned}
$$

Thus, recalling our definition of parametric surface integral, we obtain

$$
\begin{aligned}
\int\!\!\int_{\mathbf{P}} \mathbf{F} \cdot d\mathbf{A} &= \int\!\!\int_{(0,1)\times(0,1)} \left\{ \mathbf{F}(\mathbf{P}(u,v)) \cdot [\partial_1 \mathbf{P}(u,v) \times \partial_2 \mathbf{P}(u,v)]_{\mathbf{P}(u,v)} \right\} du dv \\
&= \int\!\!\int_{(0,1)\times(0,1)} u\, du dv \\
&= \int_0^1 dv \int_0^1 u\, du \\
&= \frac{1}{2}.
\end{aligned}
$$

(b) First, we compute, for any $(u,v) \in (0,2\pi) \times (0,1)$,

$$
\begin{aligned}
\partial_1 \tau(u,v) \times \partial_2 \tau(u,v) &= (-v \sin u,\ v \cos u,\ 0) \times (\cos u,\ \sin u,\ 1) \\
&= (v \cos u,\ v \sin u,\ -v) \\
\mathbf{G}(\tau(u,v)) &= (v,\ v,\ v^2 \cos^2 u + v^2 \sin^2 u)_{\tau(u,v)} \\
&= (v,\ v,\ v^2)_{\tau(u,v)}.
\end{aligned}
$$

Combining the above, we then obtain

$$
\begin{aligned}
\mathbf{G}(\tau(u,v)) \cdot [\partial_1 \tau(u,v) \times \partial_2 \tau(u,v)]_{\tau(u,v)} &= (v,\ v,\ v^2) \cdot (v \cos u,\ v \sin u,\ -v) \\
&= v^2 \cos u + v^2 \sin u - v^3.
\end{aligned}
$$

Thus, by our given definition of surface integrals,

$$
\begin{aligned}
\int\!\!\int_{\tau} \mathbf{G} \cdot d\mathbf{A} &= \int\!\!\int_{(0,2\pi)\times(0,1)} \left\{ \mathbf{G}(\tau(u,v)) \cdot [\partial_1 \tau(u,v) \times \partial_2 \tau(u,v)]_{\tau(u,v)} \right\} du dv \\
&= \int_0^1 \int_0^{2\pi} (v^2 \cos u + v^2 \sin u - v^3)\, du dv
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^1 (v^2 \sin u - v^2 \cos u - v^3 u)_{u=0}^{u=2\pi} \, dv \\
&= -2\pi \int_0^1 v^3 \, dv \\
&= -\frac{\pi}{2}.
\end{aligned}
$$

(c) First, note that for any $(u, v) \times (0, 1) \times (0, 1)$, we have

$$
\begin{aligned}
\partial_1 \mathbf{q}(u, v) \times \partial_2 \mathbf{q}(u, v) &= (1,\ 0,\ 2u) \times (0,\ 1,\ 2v) \\
&= (-2u,\ -2v,\ 1), \\
\mathbf{H}(\mathbf{q}(u, v)) &= (-u,\ -v,\ u^2 + v^2)_{\mathbf{q}(u,v)}, \\
\mathbf{H}(\mathbf{q}(u, v)) \cdot [\partial_1 \mathbf{q}(u, v) \times \partial_2 \mathbf{q}(u, v)]_{\mathbf{q}(u,v)} &= (-u,\ -v,\ u^2 + v^2) \cdot (-2u,\ -2v,\ 1) \\
&= 3u^2 + 3v^2.
\end{aligned}
$$

Finally, using the above, we can evaluate the desired surface integral:

$$
\begin{aligned}
\int\int_{\mathbf{q}} \mathbf{H} \cdot d\mathbf{A} &= \int\int_{(0,1)\times(0,1)} \left\{ \mathbf{H}(\mathbf{q}(u, v)) \cdot [\partial_1 \mathbf{q}(u, v) \times \partial_2 \mathbf{q}(u, v)]_{\mathbf{q}(u,v)} \right\} du dv \\
&= \int_0^1 \int_0^1 (3u^2 + 3v^2) \, du dv \\
&= 2.
\end{aligned}
$$

(3) *(A Survey of Integration)* Let S denote the set

$$
S = \{(u, v, u^2 - v^2) \in \mathbb{R}^3 \mid (u, v) \in (0, 1) \times (0, 1)\}.
$$

(a) Show that S is a surface. In addition, give an injective parametrisation of S whose image is precisely all of S.

(b) Compute the surface integral over S of the real-valued function

$$
F : \mathbb{R}^3 \to \mathbb{R}, \qquad F(x, y, z) = xy.
$$

(The double integral you get from expanding the surface integral is not so pleasant; you will probably have to use the method of substitution twice to compute it.)

(c) Let us also assign to S the *upward-facing orientation*, i.e. the orientation in the *positive* *z-direction.* Then, compute the surface integral over S of the vector field

$$\mathbf{G}(x, y, z) = (xy^2,\ yx^2,\ 1)_{(x,y,z)}, \qquad (x, y, z) \in \mathbb{R}^3.$$

(a) S is a surface, since it is the graph of the (smooth) function

$$f : (0, 1) \times (0, 1) \to \mathbb{R}, \qquad f(u, v) = u^2 - v^2.$$

Furthermore, an injective parametrisation of all of S is given by

$$\sigma : (0, 1) \times (0, 1) \to S, \qquad \sigma(u, v) = (u,\ v,\ u^2 - v^2).$$

(b) We begin by computing the partial derivatives of σ:

$$\partial_1 \sigma(u, v) = (1,\ 0,\ 2u), \qquad \partial_2 \sigma(u, v) = (0,\ 1,\ -2v).$$

Taking a cross product of the above yields

$$\partial_1 \sigma(u, v) \times \partial_2 \sigma(u, v) = (-2u,\ 2v,\ 1),$$
$$|\partial_1 \sigma(u, v) \times \partial_2 \sigma(u, v)| = \sqrt{1 + 4u^2 + 4v^2}.$$

Now, by part (a), we know that σ is an injective parametrisation of S whose image is all of S. Thus, we can use σ to compute our surface integral:

$$\int\int_S F \, dA = \int\int_\sigma F \, dA.$$

To calculate the above, we first note that

$$F(\sigma(u, v)) = F(u, v, u^2 - v^2) = uv.$$

Thus, our surface integral can now be expanded as

$$\int\int_S F \, dA = \int\int_{(0,1)\times(0,1)} \left(uv\sqrt{1 + 4u^2 + 4v^2} \right) du dv.$$

This can now be evaluated using Fubini's theorem and the method of subsitution:

$$
\begin{aligned}
\int\int_S F\,dA &= \int_0^1 v\left[\int_0^1 u\sqrt{1+4u^2+4v^2}\,du\right] dv \\
&= \int_0^1 v\left[\frac{1}{12}(1+4u^2+4v^2)^{\frac{3}{2}}\right]_{u=0}^{u=1} dv \\
&= \frac{1}{12}\int_0^1 v\left[(5+4v^2)^{\frac{3}{2}} - (1+4v^2)^{\frac{3}{2}}\right] dv \\
&= \frac{1}{12}\cdot\frac{1}{20}\cdot\left[(5+4v^2)^{\frac{5}{2}} - (1+4v^2)^{\frac{5}{2}}\right]_{v=0}^{v=1} \\
&= \frac{1}{240}\left(9^{\frac{5}{2}} - 5^{\frac{5}{2}} - 5^{\frac{5}{2}} + 1^{\frac{5}{2}}\right) \\
&= \frac{61}{60} - \frac{5\sqrt{5}}{24}. \qquad \text{(Sorry ☺)}
\end{aligned}
$$

(Even if you were not able to get the final number, the most important part is that you can correctly expand the surface integral into a double integral.)

(c) First, recall from part (b) that

$$\partial_1\sigma(u,v)\times\partial_2\sigma(u,v) = (-2u,\ 2v,\ 1),$$

hence it follows that σ generates the upward-facing orientation of S. Consequently, we can use the parametrisation σ to compute our surface integrals:

$$\int\int_S \mathbf{G}\cdot d\mathbf{A} = +\int\int_U \{\mathbf{G}(\sigma(u,v))\cdot[\partial_1\sigma(u,v)\times\partial_2\sigma(u,v)]_{\sigma(u,v)}\}\,du dv.$$

To calculate the above, we observe that

$$\mathbf{G}(\sigma(u,v)) = (uv^2,\ vu^2,\ 1)_{\sigma(u,v)},$$

and hence

$$\mathbf{G}(\sigma(u,v))\cdot[\partial_1\sigma(u,v)\times\partial_2\sigma(u,v)]_{\sigma(u,v)} = (uv^2,\ vu^2,\ 1)\cdot(-2u,\ 2v,\ 1) = 1.$$

Therefore, we conclude that

$$\int\int_S \mathbf{G}\cdot d\mathbf{A} = \int\int_{(0,1)\times(0,1)} 1\,du dv = 1.$$

(4) **[Marked]** Let C denote the following disconnected surface:

$$C = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1 \cup x^2 + y^2 = 4, \ -1 < z < 1\}.$$

which describes a small cylinder surrounded by a larger cylinder. Moreover, let us orient $\mathcal{C}$ such that the outer cylinder has *outward* orientation and the inner cylinder is oriented *inwards*.

(a) Sketch this surface.

(b) Compute the surface integral over C of the function

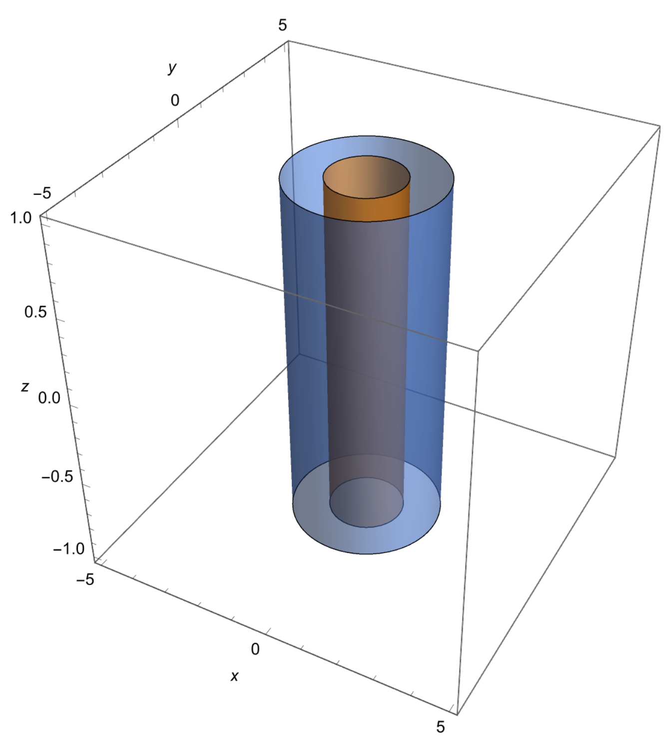
$$G : \mathbb{R}^3 \to \mathbb{R}, \qquad G(x, y, z) = -\frac{x^2y^2}{66} + \frac{z^2}{8}.$$

Some useful hints: $2\sin(x)\cos(x) = \sin(2x)$ *and* $\sin^2(x) = \frac{1-\cos(2x)}{2}$.

(c) Compute the surface integral over C of the vector field $\mathbf{H}$ on $\mathbb{R}^3$ given by

$$\mathbf{H}(x, y, z) = (y, x, z)_{(x,y,z)},$$

(a) A sketch of this figure is shown here



[1 mark for somewhat correct sketch]

(b) The first step is to parametrise C appropriately. Since this surface is disconnected, we need two parametrizations. The outer cylinder has outward orientation, so for this portion of the surface we will parametrize it as follows:

$$\sigma_{\text{out}} : (0, 2\pi) \times (-1, 1) \to C, \qquad \sigma_{\text{out}}(u, v) = (2\cos u,\ 2\sin u,\ v).$$

For the inner cylinder, we will choose to parametrize such that the parametrization aligns with the orientation of the surface, thus we choose:

$$\sigma_{\text{in}} : (0, 2\pi) \times (-1, 1) \to C, \qquad \sigma_{\text{in}}(u, v) = (\cos u,\ -\sin u,\ v).$$

Observe $\sigma_{\text{in}} \cup \sigma_{\text{out}}$ is injective, and its image is C up to a pair of lines. [2 marks for correct parametrisation]

For the outer cylinder, we compute

$$\begin{aligned}
\partial_1 \sigma_{\text{out}}(u, v) &= (-2\sin u,\ 2\cos u,\ 0),\\
\partial_2 \sigma_{\text{out}}(u, v) &= (0,\ 0,\ 1),\\
\partial_1 \sigma_{\text{out}}(u, v) \times \partial_2 \sigma_{\text{out}}(u, v) &= (2\cos u,\ 2\sin u,\ 0),\\
|\partial_1 \sigma_{\text{out}}(u, v) \times \partial_2 \sigma_{\text{out}}(u, v)| &= 2.
\end{aligned}$$

Note that this parametrisation generates the orientation of C along the outer cylinder.

Similarly, for the inner cylinder we compute:

$$\begin{aligned}
\partial_1 \sigma_{\text{in}}(u, v) &= (-\sin u,\ -\cos u,\ 0),\\
\partial_2 \sigma_{\text{in}}(u, v) &= (0,\ 0,\ 1),\\
\partial_1 \sigma_{\text{in}}(u, v) \times \partial_2 \sigma_{\text{in}}(u, v) &= (-\cos u,\ \sin u,\ 0),\\
|\partial_1 \sigma_{\text{in}}(u, v) \times \partial_2 \sigma_{\text{in}}(u, v)| &= 1.
\end{aligned}$$

and again this parametrisation points inward along the inner cylinder, as needed. [2 marks for correct calculations up to here][1 mark for correct observation of orientation]

Also we compute

$$G(\sigma_{\text{out}}(u, v)) |\partial_1 \sigma_{\text{out}}(u, v) \times \partial_2 \sigma_{\text{out}}(u, v)| = \frac{v^2}{4} - \frac{16}{33} \sin^2 u \cos^2 u\ ,$$

$$= \frac{v^2}{4} - \frac{4}{33}\sin^2(2u) \ ,$$
$$= \frac{v^2}{4} - \frac{2}{33}(1-\cos(4u)) \ .$$

A similar computation yields

$$G(\sigma_{\text{in}})|\partial_1\sigma_{\text{in}}(u,v) \times \partial_2\sigma_{\text{in}}(u,v)| = \frac{v^2}{8} - \frac{1}{528}(1-\cos(4u)) \ .$$

[1 mark for correct evalution]

We can nowcompute the surface integral over C:

$$\int\int_C G\, dA = \int\int_{\sigma_{\text{in}}} G\, dA + \int\int_{\sigma_{\text{out}}} dA$$
$$= \int\int_{(0,2\pi)\times(-1,1)} [G(\sigma_{\text{in}}(u,v))|\partial_1\sigma_{\text{in}}(u,v) \times \partial_2\sigma_{\text{in}}(u,v)| + G(\sigma_{\text{out}}(u,v))|\partial_1\sigma_{\text{out}}(u,v) \times \partial_2\sigma_{\text{out}}(u,v)|]\ dudv$$
$$= \int_0^{2\pi} du \int_{-1}^1 dv \left(\frac{3v^2}{8} - \frac{1}{16} + \frac{1}{16}\cos(4u)\right) .$$

[1 mark for almost correct answer up to this point] From here, we directly compute

$$\int\int_C G\, dA = \int_0^{2\pi} \left(\frac{v^3}{8} - \frac{v}{16} + \frac{v}{16}\cos(4u)\right)\Big|_{v=-1}^{v=1} du$$
$$= \int_0^{2\pi} \left(\frac{1}{4} - \frac{1}{8} + \frac{1}{8}\cos(4u)\right) du$$
$$= \left(\frac{u}{4} - \frac{u}{8} + \frac{1}{32}\sin(4u)\right)\Big|_{u=0}^{u=2\pi}$$
$$= \frac{2\pi}{4} - \frac{2\pi}{8} = \frac{\pi}{4}.$$

[1 mark for somewhat correct integral]

(c) We have already done most of the work, as can again use the parametrisation $\sigma_{\text{in/out}}$ from (a). Recall that our parametrisations for the inner and outer cylinders are aligned with the orientation of C, therefore we simply need to compute

$$\mathbf{H}(\sigma_{\text{in}}(u,v)) \cdot [\partial_1\sigma_{\text{in}}(u,v) \times \partial_2\sigma_{\text{in}}(u,v)] = \sin(2u)$$

and

$$\mathbf{H}(\sigma_{\text{out}}(u,v)) \cdot [\partial_1\sigma_{\text{out}}(u,v) \times \partial_2\sigma_{\text{out}}(u,v)] = 4\sin(2u)$$

[1 mark for computation]

$$\int\int_C \mathbf{H}\cdot d\mathbf{A} =$$
$$\int\int_{(0,2\pi)\times(-1,1)} \{\mathbf{H}(\sigma_{\text{in}}(u,v))\cdot[\partial_1\sigma_{\text{in}}(u,v)\times\partial_2\sigma_{\text{in}}(u,v)]_{\sigma_{\text{in}}(u,v)}\}\,dudv$$
$$+\int\int_{(0,2\pi)\times(-1,1)} \{\mathbf{H}(\sigma_{\text{out}}(u,v))\cdot[\partial_1\sigma_{\text{out}}(u,v)\times\partial_2\sigma_{\text{out}}(u,v)]_{\sigma_{\text{out}}(u,v)}\}\,dudv.$$

The integral can now be computed directly:

$$\begin{aligned}\int\int_C \mathbf{H}\cdot d\mathbf{A} &= +\int\int_{(0,2\pi)\times(-1,1)} 5\sin(2u)\,dudv\\ &= \int_0^{2\pi} 10\sin(2u)\,dudv\\ &= -5\cos(2u)\Big|_{u=0}^{u=2\pi}\\ &= 0.\end{aligned}$$

[1 mark for an almost correct answer]

(5) [Tutorial]

(a) Consider the surface *(you may assume this is indeed a surface)*

$$\mathcal{P} = \{(u,v,u^4+v)\in\mathbb{R}^3 \mid (u,v)\in(0,1)\times(-1,1)\}.$$

Compute the surface integral over $\mathcal{P}$ of the following function:

$$F:\mathbb{R}^3\to\mathbb{R},\qquad F(x,y,z)=6x^5.$$

(b) Consider the sphere,

$$\mathbb{S}^2 = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2=1\},$$

and let $\mathbb{S}^2$ be given the "outward-facing" orientation. Compute the surface integral over $\mathbb{S}^2$ of the vector field $\mathbf{F}$ on $\mathbb{R}^3$ defined by the formula

$$\mathbf{F}(x,y,z) = (0,0,z^3)_{(x,y,z)}.$$

(a) The first step is to appropriately parametrise $\mathcal{P}$. Observe that the map

$$\sigma : (0,1) \times (-1,1) \to \mathcal{P}, \qquad \sigma(u,v) = (u,\, v,\, u^4 + v)$$

is a parametrisation of $\mathcal{P}$. Moreover, note that σ is injective, and its image is all of $\mathcal{P}$. As a result, we have, from the definition of surface integrals,

$$\int\int_{\mathcal{P}} F\, dA = \int\int_{\sigma} F\, dA = \int\int_{(0,1)\times(-1,1)} F(\sigma(u,v)) |\partial_1 \sigma(u,v) \times \partial_2 \sigma(u,v)|\, du dv.$$

Next, the partial derivatives of σ satisfy

$$\partial_1 \sigma(u,v) = (1,\, 0,\, 4u^3), \qquad \partial_2 \sigma(u,v) = (0,\, 1,\, 1).$$

Thus, the required terms in the above integrand satisfy

$$\begin{aligned} |\partial_1 \sigma(u,v) \times \partial_2 \sigma(u,v)| &= |(-4u^3,\, -1,\, 1)| = \sqrt{2 + 16u^6}, \\ F(\sigma(u,v)) &= 6u^5. \end{aligned}$$

Combining all the above, we can now compute the surface integral as

$$\begin{aligned} \int\int_{\mathcal{P}} F\, dA &= \int_{-1}^{1} \int_0^1 6u^5 \sqrt{2 + 16u^6}\, du dv \\ &= 2 \int_0^1 6u^5 \sqrt{2 + 16u^6}\, du \\ &= 2 \cdot \frac{1}{16} \cdot \frac{2}{3} \cdot \left[(2 + 16u^6)^{\frac{3}{2}} \right]_{u=0}^{u=1} \\ &= \frac{13\sqrt{2}}{3}. \end{aligned}$$

(b) Recall (from lectures and the lecture notes) that the parametrisation of $\mathbb{S}^2$ given by

$$\rho : (0, 2\pi) \times (0, \pi) \to \mathbb{S}^2, \qquad \rho(u,v) = (\cos u \sin v,\, \sin u \sin v,\, \cos v),$$

is injective, and that its image is "almost all" of $\mathbb{S}^2$ (the image excludes only two points and a semicircle). Moreover, from the usual computations, we have that

$$\partial_1 \rho(u,v) \times \partial_2 \rho(u,v) = -\sin v \cdot (\cos u \sin v,\, \sin u \sin v,\, \cos v) = -\sin v \cdot \rho(u,v).$$

In particular, the arrows

$$[\partial_1 \rho(u,v) \times \partial_2 \rho(u,v)]_{\rho(u,v)} = -\sin v \cdot \rho(u,v)_{\rho(u,v)},$$

which are normal to $\mathbb{S}^2$, point inward from $\mathbb{S}^2$. Thus, ρ generates the orientation opposite to our given orientation of $\mathbb{S}^2$, and hence we have that

$$\iint_{\mathbb{S}^2} \mathbf{F} \cdot \mathbf{dA} = -\iint_{(0,2\pi)\times(0,\pi)} \left\{ \mathbf{F}(\rho(u,v)) \cdot [\partial_1 \rho(u,v) \times \partial_2 \rho(u,v)]_{\rho(u,v)} \right\} du dv.$$

Note the integrand satisfies

$$\begin{aligned} \mathbf{F}(\rho(u,v)) \cdot [\partial_1 \rho(u,v) \times \partial_2 \rho(u,v)]_{\rho(u,v)} &= -\sin v (0,\, 0,\, \cos^3 v) \cdot (\cos u \sin v,\, \sin u \sin v,\, \cos v) \\ &= -\sin v \cos^4 v. \end{aligned}$$

As a result,

$$\begin{aligned} \iint_{\mathbb{S}^2} \mathbf{F} \cdot \mathbf{dA} &= \int_0^{2\pi} \int_0^{\pi} \sin v \cos^4 v \, dv du \\ &= 2\pi \cdot \frac{1}{5} [-\cos^5 v]_{v=0}^{v=\pi} \\ &= \frac{4\pi}{5}. \end{aligned}$$

(6) *(A-levels, revisited)*

(a) Show that the surface area of a *sphere of radius* $r > 0$,

$$S_r = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = r^2\},$$

is equal to $4\pi r^2$.

(b) Show that the area of the side of a cone with base radius $r > 0$ and height $h > 0$,

$$C_{r,h} = \left\{ (x,y,z) \in \mathbb{R}^3 \mid 0 < z < h,\; x^2 + y^2 = r^2 \left(1 - \frac{z}{h}\right)^2 \right\},$$

is equal to $\pi r \sqrt{r^2 + h^2}$.

(a) Similar to the case of a unit sphere, we see that

$$\rho_r : (0, 2\pi) \times (0, \pi) \to S_r, \qquad \rho_r(u, v) = (r \cos u \sin v, r \sin u \sin v, r \cos v)$$

is an injective parametrisation of S_r, whose image is all of S_r except for two points and a semicircle. Moreover, a direct calculation (analogous to the one for $\mathbb{S}^2$) shows that

$$|\partial_1 \rho_r(u, v) \times \partial_2 \rho_r(u, v)| = |-r \sin v \cdot \rho_r(u, v)| = r^2 \sin v.$$

As a result, we obtain

$$\mathcal{A}(S_r) = \int\int_{(0,2\pi)\times(0,\pi)} r^2 \sin v \, du dv = r^2 \int_0^{2\pi} du \int_0^{\pi} \sin v \, dv = 4\pi r^2.$$

(b) The main step is to parametrise $C_{r,h}$ correctly. For this, we can take

$$\sigma : (0, 2\pi) \times (0, h) \to C_{r,h}, \qquad \sigma(u, v) = (r(1 - vh^{-1}) \cos u, r(1 - vh^{-1}) \sin u, v).$$

In particular, σ is injective, and its image is all of $C_{r,h}$ except for a line. (Plot this out and see for yourself!) Moreover, direct computations yield

$$\begin{aligned}
\partial_1 \sigma(u, v) &= (-r(1 - vh^{-1}) \sin u, \; r(1 - vh^{-1}) \cos u, \; 0), \\
\partial_2 \sigma(u, v) &= (-rh^{-1} \cos u, \; -rh^{-1} \sin u, \; 1), \\
\partial_1 \sigma(u, v) \times \partial_2 \sigma(u, v) &= (r(1 - vh^{-1}) \cos u, \; r(1 - vh^{-1}) \sin u, \; r^2 h^{-1}(1 - vh^{-1})), \\
|\partial_1 \sigma(u, v) \times \partial_2 \sigma(u, v)| &= r \left(1 - \frac{v}{h}\right) \sqrt{1 + \left(\frac{r}{h}\right)^2}.
\end{aligned}$$

Combining the above, we conclude that the surface area is

$$\begin{aligned}
\mathcal{A}(C_{r,h}) &= r \sqrt{1 + \left(\frac{r}{h}\right)^2} \int\int_{(0,2\pi)\times(0,h)} \left(1 - \frac{v}{h}\right) du dv \\
&= 2\pi r \sqrt{1 + \left(\frac{r}{h}\right)^2} \int_0^h \left(1 - \frac{v}{h}\right) dv \\
&= 2\pi r \sqrt{1 + \left(\frac{r}{h}\right)^2} \cdot \frac{h}{2} \\
&= \pi r \sqrt{r^2 + h^2}.
\end{aligned}$$

(7) *(Reversal of orientations)* Let $S \subseteq \mathbb{R}^3$ be an oriented surface, and let $\sigma : U \to S$ be a parametrisation of S. Moreover, define the set

$$U_r = \{(v, u) \mid (u, v) \in U\}$$

and define the parametric surface

$$\sigma_r : U_r \to \mathbb{R}^3, \qquad \sigma_r(v, u) = \sigma(u, v).$$

In other words, σ_r is precisely σ but with the roles of u and v reversed.

(a) Show that for any $(u, v) \in U$,

$$\partial_1 \sigma_r(v, u) \times \partial_2 \sigma_r(v, u) = -[\partial_1 \sigma(u, v) \times \partial_2 \sigma(u, v)].$$

(b) Show that σ_r is also a parametrisation of S, and that σ_r has the same image as σ.

(c) Use the formula from part (a) to conclude that if σ generates an orientation O of S, then σ_r generates the orientation opposite to O.

(a) We begin by relating the partial derivatives of σ and σ_r—for any $(v, u) \in U_r$,

$$\partial_1 \sigma_r(v, u) = \partial_v[\sigma_r(v, u)] = \partial_v[\sigma(u, v)] = \partial_2 \sigma(u, v),$$
$$\partial_2 \sigma_r(v, u) = \partial_u[\sigma_r(v, u)] = \partial_u[\sigma(u, v)] = \partial_1 \sigma(u, v).$$

As a result, using that the cross product is antisymmetric, we conclude that

$$\begin{aligned} \partial_1 \sigma_r(v, u) \times \partial_2 \sigma_r(v, u) &= \partial_2 \sigma(u, v) \times \partial_1 \sigma(u, v) \\ &= -[\partial_1 \sigma(u, v) \times \partial_2 \sigma(u, v)]. \end{aligned}$$

(b) First, suppose $\mathbf{p}$ is in the image of σ, so that $\mathbf{p} = \sigma(u, v)$ for some $(u, v) \in U$. Then, by definition, $(v, u) \in U_r$ and $\sigma_r(v, u) = \sigma(u, v) = \mathbf{p}$, and it follows that $\mathbf{p}$ is also in the image of σ_r. Conversely, if $\mathbf{p}$ is in the image of σ_r, then $\mathbf{p} = \sigma_r(v, u)$ for some $(v, u) \in U_r$. This then implies $(u, v) \in U$ and $\sigma(u, v) = \sigma_r(v, u) = \mathbf{p}$, and hence $\mathbf{p}$ is also in the image of σ_r. From the above, we conclude that σ and σ_r have the same image.

In particular, the above implies that the image of σ_r lies within S. Moreover, using the formula obtained from part (a), we have, for any $(v, u) \in U_r$,

$$|\partial_1 \sigma_r(v, u) \times \partial_2 \sigma_r(v, u)| = |\partial_1 \sigma(u, v) \times \partial_2 \sigma(u, v)| \neq 0,$$

since σ is regular by assumption. This implies that σ_r is also regular.

Combining the above, we conclude that σ_r is indeed a parametrisation of S.

(c) For any point $\mathbf{p} = \sigma(u, v) = \sigma_r(v, u)$ of S (where $(u, v) \in U$), we have that:

- The orientation generated by σ at $\mathbf{p}$ is given by

$$\mathbf{n}_\sigma(u, v) = + \left[\frac{\partial_1 \sigma(u, v) \times \partial_2 \sigma(u, v)}{|\partial_1 \sigma(u, v) \times \partial_2 \sigma(u, v)|} \right]_{\sigma(u,v)}.$$

- Recalling the result from (a), the orientation selected by σ_r at $\mathbf{p}$ is given by

$$\begin{aligned} \mathbf{n}_{\sigma_r}(v, u) &= + \left[\frac{\partial_1 \sigma_r(v, u) \times \partial_2 \sigma_r(v, u)}{|\partial_1 \sigma_r(v, u) \times \partial_2 \sigma_r(v, u)|} \right]_{\sigma_r(v,u)} \\ &= - \left[\frac{\partial_1 \sigma(u, v) \times \partial_2 \sigma(u, v)}{|\partial_1 \sigma(u, v) \times \partial_2 \sigma(u, v)|} \right]_{\sigma(u,v)} \\ &= -\mathbf{n}_\sigma(u, v). \end{aligned}$$

In particular, the above shows that σ_r generates the opposite unit normals as σ, and hence σ_r generates the orientation opposite to that of σ.

(8) *(The paradox of Gabriel's horn)* Consider the surface of revolution

$$G = \left\{ (x, y, z) \in \mathbb{R}^3 \,\middle|\, y^2 + z^2 = \frac{1}{x^2},\ x > 1 \right\},$$

which is sometimes nicknamed *Gabriel's horn*. *(Before proceeding, you should search for "Gabriel's horn" on Google Images to see an illustration of* G*.)*

(a) Show that G has infinite surface area.

(b) Show that the interior of G,

$$I = \left\{ (x, y, z) \in \mathbb{R}^3 \,\middle|\, y^2 + z^2 \leq \frac{1}{x^2},\ x > 1 \right\},$$

has finite volume.

In other words, *you can fill up the inside of the "horn" with a finite amount of paint, but you cannot paint the "horn" itself using a finite amount of paint*!

(a) To compute the surface area, we first parametrise G appropriately:

$$\sigma : (1, \infty) \times (0, 2\pi) \to G, \qquad \sigma(u, v) = (u,\ u^{-1} \cos v,\ u^{-1} \sin v).$$

Note in particular that σ is injective, and its image is all of G except for a curve. (The reasoning here is analogous to that for Question (4).)

Next, we do some computations involving σ:

$$\begin{aligned}
\partial_1 \sigma(u, v) &= (1,\ -u^{-2} \cos v,\ -u^{-2} \sin v), \\
\partial_2 \sigma(u, v) &= (0,\ -u^{-1} \sin v,\ u^{-1} \cos v), \\
\partial_1 \sigma(u, v) \times \partial_2 \sigma(u, v) &= (-u^{-3},\ -u^{-1} \cos v,\ -u^{-1} \sin v), \\
|\partial_1 \sigma(u, v) \times \partial_2 \sigma(u, v)| &= u^{-1} \sqrt{1 + u^{-4}},
\end{aligned}$$

Combining the above with the definition of surface area, we conclude that

$$\begin{aligned}
A(G) &= \int\int_{(1,\infty)\times(0,2\pi)} |\partial_1 \sigma(u, v) \times \partial_2 \sigma(u, v)|\, du dv \\
&= \int_0^{2\pi} dv \int_1^{\infty} \frac{1}{u} \sqrt{1 + \frac{1}{u^4}}\, du.
\end{aligned}$$

Since $1 + u^{-4} \geq 1$ for all $u \in \mathbb{R}$, it follows that

$$A(G) \geq 2\pi \int_1^{\infty} \frac{1}{u}\, du = \lim_{u \nearrow \infty} \ln u - \ln 1 = +\infty.$$

Thus, we conclude that $A(G)$ is indeed infinite.

(b) Recall the volume of I is

$$V(I) = \iiint_I 1\, dx dy dz.$$

The easiest way to describe I in a way that is convenient for integration is to do a change of variables and write y and z in terms of polar coordinates:

$$x = x, \qquad y = r \cos \theta, \qquad z = r \sin \theta.$$

In particular, I can be described in these new coordinates as

$$I = \{(x, r, \theta) \in \mathbb{R}^3 \mid x > 1,\ 0 \le r \le x^{-1},\ 0 \le \theta \le 2\pi\}.$$

Note that the Jacobian with respect to this change of variables is

$$J = \det \frac{\partial(x, y, z)}{\partial(x, r, \theta)} = \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -r\sin\theta \\ 0 & \sin\theta & r\cos\theta \end{bmatrix} = r.$$

Thus, by the change of variables formula and Fubini's theorem, we have that

$$\begin{aligned} V(I) &= \int_0^{2\pi} \int_1^{\infty} \int_0^{x^{-1}} J\, dr dx d\theta \\ &= \int_0^{2\pi} d\theta \int_1^{\infty} \int_0^{x^{-1}} r\, dr dx \\ &= 2\pi \cdot \frac{1}{2} \int_1^{\infty} \frac{1}{x^2}\, dx \\ &= \pi. \end{aligned}$$

Thus, the volume of I is indeed finite.