MTH5113 (2023/24): Problem Sheet 8
Solutions

---

(1) *(Warm-up)* For each of the following parts:

(i) Sketch the surface S.

(ii) Draw the unit normal $\mathbf{n_p}$ on the sketch from part (i).

(iii) Give an informal description (e.g. "outward-facing", "inward-facing", "upward-facing") of the side of S represented by the normal $\mathbf{n_p}$.

(a) $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$, and

$$\mathbf{n_p} = (0, 1, 0)_{(0,-1,0)}.$$

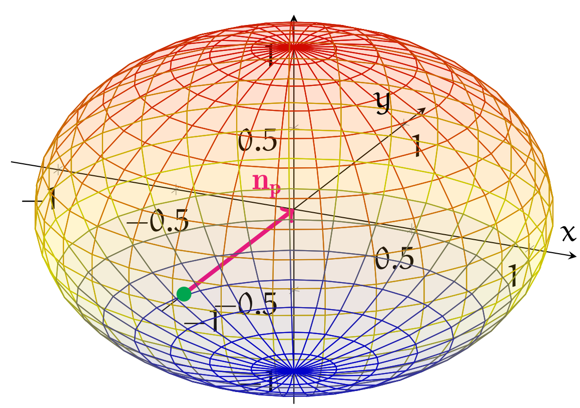
(b) $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$, and

$$\mathbf{n_p} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)_{\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{3}{4}\right)}.$$

(c) $S = \{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 + y^2\}$, and

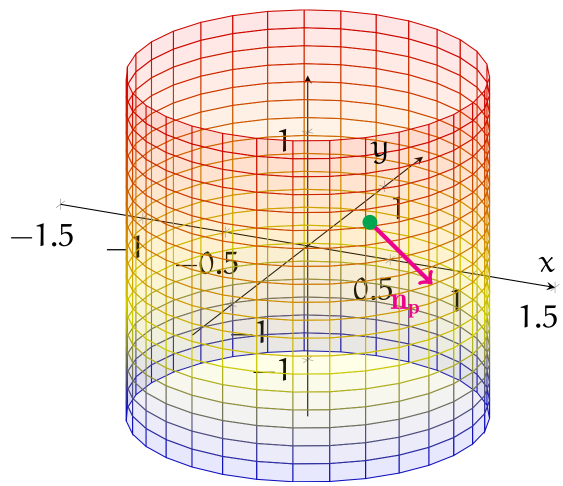
$$\mathbf{n_p} = \left(\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}\right)_{(1,-1,2)}.$$

(a) The sketch of S and $\mathbf{n_p}$ is given below:



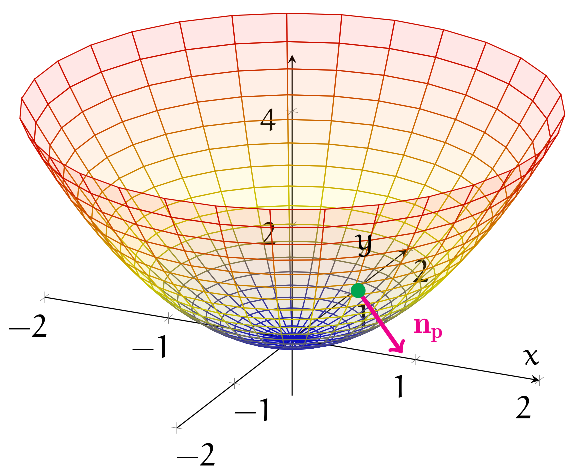
The unit normal $\mathbf{n_p}$ represents the "inward-facing" orientation of S.

(b) The sketch of S and $\mathbf{n_p}$ is given below:



The unit normal $\mathbf{n_p}$ represents the "outward-facing" orientation of S.

(c) The sketch of S and $\mathbf{n_p}$ is given below:



$\mathbf{n_p}$ represents the "outward"/"downward" (i.e. decreasing z-value) orientation of S.

(2) *(Warm-up)* Compute the surface areas of the following parametric surfaces:

(a) *Parametric torus:*

$$\alpha : (0, 2\pi) \times (0, 2\pi) \to \mathbb{R}^3, \qquad \alpha(u, v) = ((2 + \cos u) \cos v,\ (2 + \cos u) \sin v,\ \sin u).$$

(See Question (8b) of Problem Sheet 1 for a plot of α.)

(b) *Parallellogram spanned by vectors* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$*:*

$$\beta : (0,1) \times (0,1) \to \mathbb{R}^3, \qquad \beta(u,v) = u \cdot \mathbf{a} + v \cdot \mathbf{b}.$$

State your answer in terms of $\mathbf{a}$ and $\mathbf{b}$.

(a) By direct computations (see also Question (1) of Problem Sheet 7), we obtain

$$\begin{aligned} \partial_1 \sigma(u,v) &= (-\sin u \cos v,\ -\sin u \sin v,\ \cos u), \\ \partial_2 \sigma(u,v) &= (-(2+\cos u)\sin v,\ (2+\cos u)\cos v,\ 0), \\ \partial_1 \sigma(u,v) \times \partial_2 \sigma(u,v) &= -(2+\cos u) \cdot (\cos u \cos v,\ \cos u \sin v,\ \sin u) \\ |\partial_1 \sigma(u,v) \times \partial_2 \sigma(u,v)| &= |2+\cos u| \sqrt{\cos^2 u \cos^2 v + \cos^2 u \sin^2 v + \sin^2 v} \\ &= 2 + \cos u. \end{aligned}$$

Thus, by the definition of (parametric) surface area, we conclude that

$$\begin{aligned} \mathcal{A}(\alpha) &= \iint_{(0,2\pi)\times(0,2\pi)} |\partial_1 \alpha(u,v) \times \partial_2 \alpha(u,v)|\, du dv \\ &= \int_0^{2\pi} \int_0^{2\pi} (2+\cos u)\, du dv \\ &= 2\pi \int_0^{2\pi} 2\, du \\ &= 8\pi^2. \end{aligned}$$

(In the above, we applied Fubini's theorem and the fundamental theorem of calculus.)

(b) The first step is to take partial derivatives of β. In order to do this carefully, we expand $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ and then compute

$$\begin{aligned} \beta(u,v) &= (ua_1 + vb_1,\ ua_2 + vb_2,\ ua_3 + vb_3), \\ \partial_1 \beta(u,v) &= (a_1, a_2, a_3) = \mathbf{a}, \\ \partial_2 \beta(u,v) &= (b_1, b_2, b_3) = \mathbf{b}. \end{aligned}$$

In particular, we note that

$$|\partial_1 \beta(u,v) \times \partial_2 \beta(u,v)| = |\mathbf{a} \times \mathbf{b}|,$$

which is a constant. As a result, we obtain, for the surface area,

$$\begin{aligned}
\mathcal{A}(\beta) &= \iint_{(0,1)\times(0,1)} |\partial_1\beta(u,v) \times \partial_2\beta(u,v)| \, du dv \\
&= \int_0^1 \int_0^1 |\mathbf{a} \times \mathbf{b}| \, du dv \\
&= |\mathbf{a} \times \mathbf{b}|.
\end{aligned}$$

(3) [**Marked**] Consider the surface of revolution

$$\mathcal{H} = \left\{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + z^2 = \left(\frac{3}{2} + \cos \frac{y}{2} \right)^2 \right\}.$$

(a) Find the tangent plane to $\mathcal{H}$ at $\left(\frac{3}{2\sqrt{2}}, \pi, -\frac{3}{2\sqrt{2}}\right)$. *(See also (Q7) of Problem Sheet 7.)*

(b) Find the unit normals to $\mathcal{H}$ at $\left(\frac{3}{2\sqrt{2}}, \pi, -\frac{3}{2\sqrt{2}}\right)$.

(c) Which of the two unit normals in (b) represents the "outward-facing" side of $\mathcal{H}$?

(For part (c), you do not have to prove the answer. You can find the answer by sketching $\mathcal{H}$ and the appropriate normals and then inspecting your sketch.)

(a) The first step is to give a parametrisation of $\mathcal{H}$ that passes through $\left(\frac{3}{2\sqrt{2}}, \pi, -\frac{3}{2\sqrt{2}}\right)$. The easiest option is to consider a modification of the method used in question (7) of Problem Sheet 7 and use polar coordinates around the y-axis:

$$\sigma : \mathbb{R}^2 \to \mathcal{H}, \qquad \sigma(u, v) = \left(\left(\frac{3}{2} + \cos \frac{u}{2} \right) \cos v, \, u, \, \left(\frac{3}{2} + \cos \frac{u}{2} \right) \sin v \right).$$

The partial derivatives of σ are then given by:

$$\begin{aligned}
\partial_1\sigma(u,v) &= \left(-\frac{1}{2} \sin \frac{u}{2} \cos v, \, 1, \, -\frac{1}{2} \sin \frac{u}{2} \sin v \right), \\
\partial_2\sigma(u,v) &= \left(-\left(\frac{3}{2} + \cos \frac{u}{2} \right) \sin v, \, 0, \, \left(\frac{3}{2} + \cos \frac{u}{2} \right) \cos v \right).
\end{aligned}$$

[1 mark for an almost correct answer up to this point]

Note that $\left(\frac{3}{2\sqrt{2}}, \pi, -\frac{3}{2\sqrt{2}}\right) = \sigma(\pi, -\frac{\pi}{4})$. Thus, evaluating at $(u, v) = (\pi, -\frac{\pi}{4})$, we obtain

$$\partial_1\sigma\left(\pi, -\frac{\pi}{4}\right) = \left(-\frac{1}{2\sqrt{2}}, 1, \frac{1}{2\sqrt{2}}\right), \qquad \partial_2\sigma\left(\pi, -\frac{\pi}{4}\right) = \left(\frac{3}{2\sqrt{2}}, 0, \frac{3}{2\sqrt{2}}\right).$$

As a result, the tangent plane to $\mathcal{H}$ at $(\sqrt{2}, \pi, -\sqrt{2})$ is given by

$$\begin{aligned} T_{(\sqrt{2},\pi,-\sqrt{2})}\mathcal{H} &= T_\sigma\left(\pi, -\frac{\pi}{4}\right) \\ &= \left\{ a \cdot \left(-\frac{1}{2\sqrt{2}}, 1, \frac{1}{2\sqrt{2}}\right)_{\left(\frac{3}{2\sqrt{2}},\pi,-\frac{3}{2\sqrt{2}}\right)} + b \cdot \left(\frac{3}{2\sqrt{2}}, 0, \frac{3}{2\sqrt{2}}\right)_{\left(\frac{3}{2\sqrt{2}},\pi,-\frac{3}{2\sqrt{2}}\right)} \middle| \, a, b \in \mathbb{R} \right\}. \end{aligned}$$

[1 mark for an almost correct answer]

(b) Taking a cross product of the partial derivatives in **(a)**, we see that

$$\begin{aligned} \partial_1\sigma\left(\pi, -\frac{\pi}{4}\right) \times \partial_2\sigma\left(\pi, -\frac{\pi}{4}\right) &= \left(\frac{3}{2\sqrt{2}}, \frac{3}{4}, -\frac{3}{2\sqrt{2}}\right), \\ \left|\partial_1\sigma\left(\pi, -\frac{\pi}{4}\right) \times \partial_2\sigma\left(\pi, -\frac{\pi}{4}\right)\right| &= \frac{3\sqrt{5}}{4}. \end{aligned}$$

As a result, the unit normals to $\mathcal{H}$ at $(\sqrt{2}, \pi, -\sqrt{2}) = \sigma(\pi, -\frac{\pi}{4})$ are given by

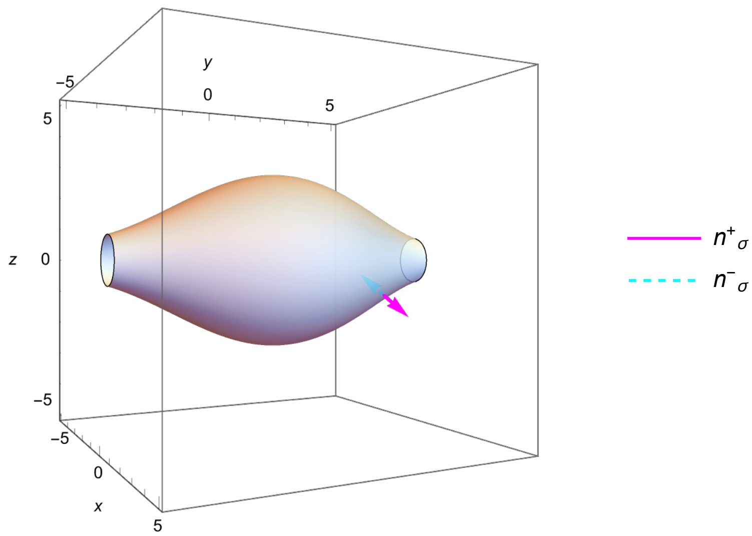
$$\begin{aligned} \mathbf{n}_\sigma^\pm\left(\pi, -\frac{\pi}{4}\right) &= \pm\left[\frac{\partial_1\sigma\left(\pi, -\frac{\pi}{4}\right) \times \partial_2\sigma\left(\pi, -\frac{\pi}{4}\right)}{\left|\partial_1\sigma\left(\pi, -\frac{\pi}{4}\right) \times \partial_2\sigma\left(\pi, -\frac{\pi}{4}\right)\right|}\right]_{\sigma(\pi,-\frac{\pi}{4})} \\ &= \pm\left(\sqrt{\frac{2}{5}}, \frac{1}{\sqrt{5}}, -\sqrt{\frac{2}{5}}\right)_{\left(\frac{3}{2\sqrt{2}},\pi,-\frac{3}{2\sqrt{2}}\right)}. \end{aligned}$$

[2 marks for an almost correct answer]

(c) The outward-facing side of $\mathcal{H}$ is captured by the unit normal

$$\mathbf{n}_\sigma^+ = +\left(\sqrt{\frac{2}{5}}, \frac{1}{\sqrt{5}}, -\sqrt{\frac{2}{5}}\right)_{\left(\frac{3}{2\sqrt{2}},\pi,-\frac{3}{2\sqrt{2}}\right)}.$$

One can see this by sketching $\mathcal{H}$ and the unit normals $\mathbf{n}_\sigma^\pm$.



[1 mark for a reasonable attempt]

(4) [**Tutorial**] Consider the sphere

$$\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

(a) Find two parametrisations of $\mathbb{S}^2$ such that the combined images of all these parametrisations cover all of $\mathbb{S}^2$.

(b) Show that the unit normals to $\mathbb{S}^2$ at any $\mathbf{p} \in \mathbb{S}^2$ are given by $\pm \mathbf{p}_{\mathbf{p}}$.

(c) What choice of unit normals of $\mathbb{S}^2$ defines the "outward-facing" orientation of $\mathbb{S}^2$? What choice of unit normals of $\mathbb{S}^2$ defines the "inward-facing" orientation of $\mathbb{S}^2$?

(a) There are many different ways to do this, and we only give one answer here. For example, one could begin with the spherical coordinate parametrisation of $\mathbb{S}^2$:

$$\rho : \mathbb{R} \times (0, \pi) \to \mathbb{R}^3, \qquad \rho(u, v) = (\cos u \sin v,\ \sin u \sin v,\ \cos v).$$

In particular, ρ is a regular parametrisation of $\mathbb{S}^2$, and its image is $\mathbb{S}^2 \setminus \{(0, 0, \pm 1)\}$—that is, all of $\mathbb{S}^2$ except for the north and south poles. (See the lecture notes for this derivativation.)

To reach the points $\{(0, 0, \pm 1)\}$ that are excluded by ρ, we can take another parametrisation that is obtained by switching around the x, y, and z-components of ρ:

$$\tau : \mathbb{R} \times (0, \pi) \to \mathbb{R}^3, \qquad \tau(u, v) = (\cos v, \cos u \sin v, \sin u \sin v).$$

From what we know about ρ, we see that the image of τ is all of $\mathbb{S}^2$ except for the points $\{(\pm 1, 0, 0)\}$. In particular, the images of ρ and τ together cover all of $\mathbb{S}^2$.

Another strategy is to use stereographic projections—see Question (8) of Problem Sheet 7.

(b) The simplest way to do this is to note that $\mathbb{S}^2$ is a level set,

$$\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid s(x, y, z) = 1\},$$

where s is the function

$$s : \mathbb{R}^3 \to \mathbb{R}, \qquad s(x, y, z) = x^2 + y^2 + z^2.$$

Taking a gradient of s yields, at each $(x, y, z) \in \mathbb{S}^2$,

$$\nabla s(x, y, z) = (2x, 2y, 2z)_{(x,y,z)}, \qquad |\nabla s(x, y, z)| = 2\sqrt{x^2 + y^2 + z^2} = 2,$$

where in the last step, we recalled that $x^2 + y^2 + z^2 = 1$ for any $(x, y, z) \in \mathbb{S}^2$. Therefore, we conclude that the unit normals to $\mathbb{S}^2$ at any $\mathbf{p} = (x, y, z) \in \mathbb{S}^2$ are

$$\mathbf{n}^{\pm}_{\mathbf{p}} = \pm \frac{1}{|\nabla s(x, y, z)|} \cdot \nabla s(x, y, z) = \pm \frac{1}{2} \cdot (2x, 2y, 2z)_{(x,y,z)} = \pm \mathbf{p}_{\mathbf{p}}.$$

The unit normals can also be computed using the parametrisations from (a).

(c) The choice of the unit normal $+\mathbf{p}_{\mathbf{p}}$ at each $\mathbf{p} \in \mathbb{S}^2$ (notice that these vary smoothly with respect to $\mathbf{p}$) defines the outward-facing orientation of $\mathbb{S}^2$. (In particular, if you plot the arrows $+\mathbf{p}_{\mathbf{p}}$ on the sphere, you will see that they all point outwards from the sphere.)

On the other hand, the choice of the unit normal $-\mathbf{p}_{\mathbf{p}}$ at each $\mathbf{p} \in \mathbb{S}^2$ defines the inward-facing orientation of $\mathbb{S}^2$. (In particular, if you plot the arrows $-\mathbf{p}_{\mathbf{p}}$ on the sphere, you will see that they all point inwards into the sphere.)

(5) *(Fun with graphs)* Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function, and let

$$G_f = \{(x, y, z) \in \mathbb{R}^3 \mid z = f(x, y)\}$$

be the graph of f, which we know to be a surface. For any $(x, y) \in \mathbb{R}^2$:

- **(a)** Find the tangent plane to G_f at $(x, y, f(x, y))$.
- **(b)** Find the unit normals to G_f at $(x, y, f(x, y))$.

Give your answers in terms of f *and its derivatives at* (x, y).

(a) First, the following is a parametrisation of G_f:

$$\sigma : \mathbb{R}^2 \to G_f, \qquad \sigma(u, v) = (u, v, f(u, v)).$$

In particular, note that $(x, y, f(x, y)) = \sigma(x, y)$.

Taking partial derivatives of σ yields

$$\partial_1\sigma(x, y) = (1, 0, \partial_1 f(x, y)), \qquad \partial_2\sigma(x, y) = (0, 1, \partial_2 f(x, y)).$$

Thus, by the definition of the tangent plane, we have

$$\begin{aligned} T_{(x,y,f(x,y))}G_f &= T_\sigma(x, y) \\ &= \left\{ a \cdot (1, 0, \partial_1 f(x, y))_{(x,y,f(x,y))} + b \cdot (0, 1, \partial_2 f(x, y))_{(x,y,f(x,y))} \,\middle|\, a, b \in \mathbb{R} \right\}. \end{aligned}$$

(b) Continuing from part (a), we compute

$$\begin{aligned} \partial_1\sigma(x, y) \times \partial_2\sigma(x, y) &= (-\partial_1 f(x, y), -\partial_2 f(x, y), 1), \\ |\partial_1\sigma(x, y) \times \partial_2\sigma(x, y)| &= \sqrt{1 + [\partial_1 f(x, y)]^2 + [\partial_2 f(x, y)]^2}. \end{aligned}$$

As a result, the unit normals are given by

$$\begin{aligned} \mathbf{n}^{\pm}_{(x,y,f(x,y))} &= \pm \left[\frac{\partial_1\sigma(x, y) \times \partial_2\sigma(x, y)}{|\partial_1\sigma(x, y) \times \partial_2\sigma(x, y)|} \right]_{\sigma(x,y)} \\ &= \pm \frac{1}{\sqrt{1 + [\partial_1 f(x, y)]^2 + [\partial_2 f(x, y)]^2}} \cdot (-\partial_1 f(x, y), -\partial_2 f(x, y), 1)_{(x,y,f(x,y))}. \end{aligned}$$

Alternatively, one can observe that G_f is the level set of the function

$$F(x, y, z) = z - f(x, y).$$

Moreover, the gradient of F satisfies

$$\nabla F(x, y, z) = (-\partial_1 f(x, y), -\partial_2 f(x, y), 1)_{(x,y,z)}.$$

Thus, the unit normals at $(x, y, f(x, y))$ are also given by

$$\begin{aligned} \mathbf{n}^{\pm}_{(x,y,f(x,y))} &= \pm \frac{1}{|\nabla F(x, y, f(x, y))|} \cdot \nabla F(x, y, f(x, y)) \\ &= \pm \frac{1}{\sqrt{1 + [\partial_1 f(x, y)]^2 + [\partial_2 f(x, y)]^2}} \cdot (-\partial_1 f(x, y), -\partial_2 f(x, y), 1)_{(x,y,f(x,y))}. \end{aligned}$$

(6) *(Tangent planes revisited)* Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function, and let

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = 0\}$$

be a level set of f. In addition, assume $\nabla f(\mathbf{p})$ is nonzero for any $\mathbf{p} \in S$, so that S is a surface. Show that at each $\mathbf{p} \in S$, the tangent plane to S at $\mathbf{p}$ satisfies

$$T_{\mathbf{p}}S = \{\mathbf{v}_{\mathbf{p}} \in T_{\mathbf{p}}\mathbb{R}^3 \mid \mathbf{v}_{\mathbf{p}} \cdot \nabla f(\mathbf{p}) = 0\}.$$

Let us denote the right-hand side of the above by V:

$$V = \{\mathbf{v}_{\mathbf{p}} \in T_{\mathbf{p}}\mathbb{R}^3 \mid \mathbf{v}_{\mathbf{p}} \cdot \nabla f(\mathbf{p}) = 0\}.$$

Since we have shown (see either the lectures or lecture notes) that $\nabla f(\mathbf{p})$ is normal to every element of $T_{\mathbf{p}}S$, it follows that $T_{\mathbf{p}}S \subseteq V$.

Next, observe that V *is a 2-dimensional vector space**. Then, since V and $T_{\mathbf{p}}S$ are both 2-dimensional subspaces of $T_{\mathbf{p}}\mathbb{R}^3$, and since $V \subseteq T_{\mathbf{p}}S$, it follows that $V = T_{\mathbf{p}}S$, as desired.

* To actually prove that V is 2-dimensional, one can do a bit of linear algebra. For this, we

let $A : T_{\mathbf{p}}\mathbb{R}^3 \to \mathbb{R}$ denote the linear operator

$$A(\mathbf{v_p}) = \mathbf{v_p} \cdot \nabla f(\mathbf{p}).$$

Note that V is the kernel, or nullspace, of A. Since A is not everywhere zero, then rank $A = 1$. Since $T_{\mathbf{p}}\mathbb{R}^3$ is 3-dimensional, we conclude that

$$\dim V = \dim(\ker A) = \dim T_{\mathbf{p}}\mathbb{R}^3 - \dim(\operatorname{rank} A) = 3 - 1 = 2.$$

(7) *(Surface area in higher dimensions)*

(a) Let $\mathcal{P}$ be a parallelogram in $\mathbb{R}^n$, with two of its sides given by tangent vectors $\mathbf{a_p}$ and $\mathbf{b_p}$ (where $\mathbf{a}, \mathbf{b}, \mathbf{p} \in \mathbb{R}^n$). Recall from lectures and the lecture notes that when $n = 3$, the area of $\mathcal{P}$ is given by $|\mathbf{a} \times \mathbf{b}|$. Show that for general n, the area of $\mathcal{P}$ satisfies

$$\mathcal{A}(\mathcal{P}) = \sqrt{\det \begin{bmatrix} \mathbf{a} \cdot \mathbf{a} & \mathbf{a} \cdot \mathbf{b} \\ \mathbf{b} \cdot \mathbf{a} & \mathbf{b} \cdot \mathbf{b} \end{bmatrix}}.$$

(In particular, when $n \neq 3$, we no longer have the cross product.)

(b) Use the results from part (a) to give a reasonable definition of the surface area of a regular parametric surface $\sigma : U \to \mathbb{R}^n$, for any dimension n.

(a) Letting $\mathbf{a_p}$ represent the "base" of $\mathcal{P}$, letting h denote the "height" of $\mathcal{P}$, and letting θ denote the angle between $\mathbf{a_p}$ and $\mathbf{b_p}$, we see (as in the lectures) that

$$\mathcal{A}(\mathcal{P}) = |\mathbf{a}| \cdot h = |\mathbf{a}||\mathbf{b}| \sin\theta.$$

Squaring the above and recalling the usual trigonometric identities, we see that

$$[\mathcal{A}(\mathcal{P})]^2 = |\mathbf{a}|^2|\mathbf{b}|^2 \sin^2\theta = |\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a}|^2|\mathbf{b}|^2 \cos^2\theta.$$

Recalling the basic properties of dot products, the above can now be written as

$$[\mathcal{A}(\mathcal{P})]^2 = (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{b})^2 = \det \begin{bmatrix} \mathbf{a} \cdot \mathbf{a} & \mathbf{a} \cdot \mathbf{b} \\ \mathbf{b} \cdot \mathbf{a} & \mathbf{b} \cdot \mathbf{b} \end{bmatrix},$$

and the desired formula follows.

(b) Recall that when $n = 3$, the definition of surface area is given by

$$\mathcal{A}(\sigma) = \iint_U |\partial_1 \sigma(u,v) \times \partial_2 \sigma(u,v)| \, du dv,$$

and the integrand $|\partial_1 \sigma(u,v) \times \partial_2 \sigma(u,v)|$ represents the area of an "infinitesimal" parallelogram at $\sigma(u,v)$. Thus, in higher dimensions, we can replace the above integrand by the corresponding formula for the area of a parallelogram in $\mathbb{R}^n$ obtained in part (a):

$$\mathcal{A}(\sigma) = \iint_U \mathcal{F}(u,v) \, du dv,$$

$$\mathcal{F}(u,v) = \sqrt{\det \begin{bmatrix} \partial_1 \sigma(u,v) \cdot \partial_1 \sigma(u,v) & \partial_1 \sigma(u,v) \cdot \partial_2 \sigma(u,v) \\ \partial_2 \sigma(u,v) \cdot \partial_1 \sigma(u,v) & \partial_2 \sigma(u,v) \cdot \partial_2 \sigma(u,v) \end{bmatrix}}.$$

(8) *(Confusion with Möbius bands)* Consider the parametric surface

$$\sigma : (-1,1) \times \mathbb{R} \to \mathbb{R}^3, \qquad \sigma(u,v) = \left(\left(1 - \frac{u}{2} \sin \frac{v}{2}\right) \cos v, \left(1 - \frac{u}{2} \sin \frac{v}{2}\right) \sin v, \frac{u}{2} \cos \frac{v}{2} \right),$$

and let M be defined as the image of σ. One can, in fact, show that M is a surface, and that σ is a parametrisation of M whose image is all of M. *(Here, you can assume both of these facts without proving them.)* In particular, this M gives an explicit description of a *Möbius band*; see Figure 4.21 in the lecture notes for an illustration of M.

Ms. Mistake (who is close friends with Mr. Error from Problem Sheet 4) decides to choose the following unit normals to M:

$$\mathbf{n}^+_\sigma(u,v) = + \left[\frac{\partial_1 \sigma(u,v) \times \partial_2 \sigma(u,v)}{|\partial_1 \sigma(u,v) \times \partial_2 \sigma(u,v)|} \right]_{\sigma(u,v)}, \qquad (u,v) \in (-1,1) \times \mathbb{R}.$$

Ms. Mistake concludes that the $\mathbf{n}^+_\sigma(u,v)$'s she chose define an orientation of M, and hence M is orientable! As a wise tutor for *MTH5113*, explain why Ms. Mistake is mistaken!

To explain this, one needs to understand how σ behaves. The first point to note that

$$\sigma(0,0) = \sigma(0,2\pi) = (1,0,0).$$

(The above is a special case of the following observation: every time the parameter v increases by 2π, the corresponding values $\sigma(0,v)$ travel one full lap around M.)

Next, let us compute the partial derivatives of σ (at $u = 0$ for simplicity):

$$\partial_1 \sigma(0, v) = \left(-\frac{1}{2} \sin \frac{v}{2} \cos v, \, -\frac{1}{2} \sin \frac{v}{2} \sin v, \, \frac{1}{2} \cos \frac{v}{2} \right),$$
$$\partial_2 \sigma(0, v) = (-\sin v, \, \cos v, \, 0).$$

Taking a cross product of the above yields

$$\partial_1 \sigma(0, v) \times \partial_2 \sigma(0, v) = -\frac{1}{2} \left(\cos v \cos \frac{v}{2}, \, \sin v \cos \frac{v}{2}, \, \sin \frac{v}{2} \right),$$
$$|\partial_1 \sigma(0, v) \times \partial_2 \sigma(0, v)| = \frac{1}{2}.$$

As a result, at $(u, v) = (0, v)$, we have

$$\mathbf{n}^+_\sigma(0, v) = -\left(\cos v \cos \frac{v}{2}, \, \sin v \cos \frac{v}{2}, \, \sin \frac{v}{2} \right)_{\sigma(0,v)}.$$

In particular, at $v = 0$ and $v = 2\pi$, we have

$$\mathbf{n}^+_\sigma(0, 0) = (-1, 0, 0)_{(1,0,0)}, \qquad \mathbf{n}^+_\sigma(0, 2\pi) = (1, 0, 0)_{(1,0,0)},$$

that is, *the* $\mathbf{n}^+_\sigma(u, v)$*'s include both unit normals of* M *at* $(1, 0, 0)$! As a result, the $\mathbf{n}^+_\sigma(u, v)$'s do *not* define an orientation of M (since an orientation is by definition a choice of *only one* unit normal at each point), hence Ms. Mistake is indeed mistaken.

(More generally, the $\mathbf{n}^+_\sigma(u, v)$'s include *both* unit normals to *any* point of M.)