MTH5113 (2023/24): Problem Sheet 7
Solutions

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(1) *(Warm-up)* For each of the parametric surfaces σ given below and every pair of parameters (u, v) in the domain of σ, compute the following:

(i) $\partial_1\sigma(u, v)$ and $\partial_2\sigma(u, v)$.

(ii) $\partial_1\sigma(u, v) \times \partial_2\sigma(u, v)$.

(iii) $|\partial_1\sigma(u, v) \times \partial_2\sigma(u, v)|$.

(a) *Sphere:* $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$, where $\sigma(u, v) = (\cos u \sin v, \ \sin u \sin v, \ \cos v)$.

(b) *Torus:* $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$, where $\sigma(u, v) = ((2 + \cos u) \cos v, \ (2 + \cos u) \sin v, \ \sin u)$.

(a) (i) Taking partial derivatives of σ yields

$$\begin{aligned} \partial_1\sigma(u, v) &= (-\sin u \sin v, \ \cos u \sin v, \ 0), \\ \partial_2\sigma(u, v) &= (\cos u \cos v, \ \sin u \cos v, \ -\sin v). \end{aligned}$$

(ii) Taking a cross product of the vectors from (i) yields

$$\begin{aligned} \partial_1\sigma(u, v) \times \partial_2\sigma(u, v) &= (-\cos u \sin^2 v - 0, \ 0 - \sin u \sin^2 v, \ -(\sin^2 u + \cos^2 u) \sin v \cos v) \\ &= -\sin v \cdot (\cos u \sin v, \ \sin u \sin v, \ \cos v). \end{aligned}$$

(iii) Taking the norm of the result from (ii) yields

$$\begin{aligned} |\partial_1\sigma(u, v) \times \partial_2\sigma(u, v)| &= |\sin v| \sqrt{\cos^2 u \sin^2 v + \sin^2 u \sin^2 v + \cos^2 v} \\ &= |\sin v| \sqrt{\sin^2 v + \cos^2 v} \\ &= |\sin v|. \end{aligned}$$

(b) (i) Taking partial derivatives of σ yields

$$\partial_1\sigma(u, v) = (-\sin u \cos v, \ -\sin u \sin v, \ \cos u),$$

$$\partial_2\sigma(u,v) = (-(2+\cos u)\sin v,\ (2+\cos u)\cos v,\ 0).$$

(ii) Taking a cross product of the vectors from (i) yields

$$\begin{aligned}\partial_1\sigma(u,v)\times\partial_2\sigma(u,v) &= -(2+\cos u)\cdot(\cos u\cos v,\ \cos u\sin v,\ \sin u(\cos^2 v+\sin^2 v))\\ &= -(2+\cos u)\cdot(\cos u\cos v,\ \cos u\sin v,\ \sin u).\end{aligned}$$

(iii) Taking the norm of the result from (ii) yields

$$\begin{aligned}|\partial_1\sigma(u,v)\times\partial_2\sigma(u,v)| &= |2+\cos u|\sqrt{\cos^2 u\cos^2 v+\cos^2 u\sin^2 v+\sin^2 v}\\ &= 2+\cos u.\end{aligned}$$

(2) *(Warm-up)* Determine whether the following parametric surfaces are regular:

(a) *Paraboloid:*

$$\sigma:\mathbb{R}^2\to\mathbb{R}^3,\qquad \sigma(u,v) = (u,\ v,\ u^2+v^2).$$

(b) *(Polar)* xy*-plane:*

$$\mathbf{P}:\mathbb{R}^2\to\mathbb{R}^3,\qquad \mathbf{P}(u,v) = (u\cos v,\ u\sin v,\ 0).$$

(c) *One-sheeted hyperboloid:*

$$\mathbf{H}:\mathbb{R}^2\to\mathbb{R}^3,\qquad \mathbf{H}(u,v) = (\cos u\cosh v,\ \sin u\cosh v,\ \sinh v).$$

(a) We begin by computing the partial derivatives of σ for any $(u,v)\in\mathbb{R}^2$:

$$\partial_1\sigma(u,v) = (1,\ 0,\ 2u),\qquad \partial_2\sigma(u,v) = (0,\ 1,\ 2v).$$

Taking the cross product of the above yields, for any $(u,v)\in\mathbb{R}^2$, that

$$\begin{aligned}\partial_1\sigma(u,v)\times\partial_2\sigma(u,v) &= (-2u,\ -2v,\ 1),\\ |\partial_1\sigma(u,v)\times\partial_2\sigma(u,v)| &= \sqrt{4u^2+4v^2+1}\geq\sqrt{1}\neq 0.\end{aligned}$$

Thus, it follows that σ is regular.

(b) Taking partial derivatives yields, for any $(u, v) \in \mathbb{R}^2$, that

$$\partial_1 \mathbf{P}(u, v) = (\cos v,\ \sin v,\ 0), \qquad \partial_2 \mathbf{P}(u, v) = (-u \sin v,\ u \cos v,\ 0).$$

Taking the cross product, we then obtain

$$\begin{aligned} \partial_1 \mathbf{P}(u, v) \times \partial_2 \mathbf{P}(u, v) &= (0,\ 0,\ u \cos^2 v + u \sin^2 v) = (0, 0, u), \\ |\partial_1 \mathbf{P}(u, v) \times \partial_2 \mathbf{P}(u, v)| &= |u|. \end{aligned}$$

In particular, the above vanishes whenever $u = 0$, hence $\mathbf{P}$ is not regular.

(c) Taking partial derivatives yields, we obtain

$$\begin{aligned} \partial_1 \mathbf{H}(u, v) &= (-\sin u \cosh v,\ \cos u \cosh v,\ 0), \\ \partial_2 \mathbf{H}(u, v) &= (\cos u \sinh v,\ \sin u \sinh v,\ \cosh v), \end{aligned}$$

for any $(u, v) \in \mathbb{R}^2$. Taking the cross product yields

$$\partial_1 \mathbf{H}(u, v) \times \partial_2 \mathbf{H}(u, v) = \cosh v \cdot (\cos u \cosh v,\ \sin u \cosh v,\ -\sinh v).$$

Taking the norm of the above, we see that

$$\begin{aligned} |\partial_1 \mathbf{H}(u, v) \times \partial_2 \mathbf{H}(u, v)| &= \cosh v \sqrt{\cos^2 u \cosh^2 v + \sin^2 u \cosh^2 v + \sinh^2 v} \\ &= \cosh v \sqrt{\cosh^2 v + \sinh^2 v} \\ &= \cosh v \sqrt{1 + 2 \sinh^2 v}, \end{aligned}$$

where we recalled the identity $\cosh^2 v - \sinh^2 v = 1$ in the last step. Finally, using that $\cosh v > 0$ and $\sinh^2 v \geq 0$ for all $v \in \mathbb{R}$, we conclude that

$$|\partial_1 \mathbf{H}(u, v) \times \partial_2 \mathbf{H}(u, v)| \geq \cosh v > 0,$$

for any $(u, v) \in \mathbb{R}^2$. As a result, $\mathbf{H}$ is regular.

(3) *(Parametrise me!)* For each surface S and point $\mathbf{p} \in S$ below, give a parametrisation σ of S such that $\mathbf{p}$ lies in the image of σ.

(a) *Plane:*

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid y = z\}, \qquad \mathbf{p} = (1, -4, -4).$$

(b) *Ellipsoid:*

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + 4y^2 + 4z^2 = 4\}, \qquad \mathbf{p} = (2, 0, 0).$$

(c) *Gabriel's Horn:*

$$S = \left\{(x, y, z) \in \mathbb{R}^3 \mid x > 0,\ y^2 + z^2 = \frac{1}{x^2}\right\}, \qquad \mathbf{p} = \left(1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

(a) One way to parametrise S is to set u to be x and v to be either y or z; the defining equation $y = z$ then implies both y and z are set to v. This leads to the parametrisation

$$\sigma : \mathbb{R}^2 \to S, \qquad \sigma(u, v) = (u, v, v).$$

One can show that σ is regular (you do not need to show this here). Moreover, σ is injective, and its image covers all of S—note that $(1, -4, -4) = \sigma(1, -4)$.

(b) The most straightforward method to set two of x, y, z to be u and v and to set the remaining component via the defining equation of S. How we make this choice is dictated by the requirement that $(2, 0, 0)$ is in the image of our parametrisation.

For example, one correct answer is to set

$$y = u, \qquad z = v, \qquad x = \sqrt{4 - 4y^2 - 4z^2} = 2\sqrt{1 - u^2 - v^2}.$$

This leads to the parametrisation,

$$\sigma : B \to S, \qquad \sigma(u, v) = \left(2\sqrt{1 - u^2 - v^2},\ u,\ v\right),$$

where $B = \{(u, v) \in \mathbb{R}^2 \mid u^2 + v^2 < 1\}$ is the unit disk about the origin. In particular, one can show that σ is regular, and that $(2, 0, 0) = \sigma(0, 0)$.

An alternative method is to rescale the usual spherical coordinate parametrisation of $\mathbb{S}^2$ (in the same way we rescaled the parametrisation of a circle to describe an ellipse). You can try it yourself—this process leads to the following parametrisation:

$$\sigma : \mathbb{R} \times (0, \pi) \to S, \qquad \sigma(u, v) = (2 \cos u \sin v, \sin u \sin v, \cos v).$$

Moreover, note that for this σ, we have $(2, 0, 0) = \sigma(0, \frac{\pi}{2})$.

(c) One natural way to parametrise is to observe that at each $x > 0$, the points of S at that x-coordinate—which satisfy $y^2 + z^2 = x^{-2}$—is a circle in the yz-plane (about the origin) of radius $\frac{1}{x}$. As a result, we can take $x = u$, and we can take v to be the polar coordinate of the circles (of radius $\frac{1}{u}$). This yields the following parametrisation:

$$\sigma : (0, \infty) \times \mathbb{R} \to S, \qquad \left(u, \frac{1}{u} \cos v, \frac{1}{u} \sin v \right).$$

Note in particular that $\mathbf{p} = \sigma(1, \frac{\pi}{2})$.

Another method is to take $x = u$ and $y = v$. Then, the defining equation for S implies

$$z^2 = \frac{1}{u^2} - v^2, \qquad z = \pm \sqrt{\frac{1}{u^2} - v^2}.$$

Since we want our parametrisation to pass through $\mathbf{p}$, which has a positive z-coordinate, we must choose the "+" sign in the above.

In addition, the above square root is only well defined when

$$\frac{1}{u^2} - v^2 > 0, \qquad v^2 < \frac{1}{u^2},$$

that is, when (u, v) lies in the (open, connected) region

$$K = \left\{ (u, v) \in \mathbb{R}^2 \mid u > 0,\, v^2 < \frac{1}{u^2} \right\}.$$

As a result, another possible parametrisation of S is

$$\tau : K \to S, \qquad \sigma(u, v) = \left(u,\, v,\, +\sqrt{\frac{1}{u^2} - v^2} \right).$$

Moreover, note that $\mathbf{p} = \tau(1, 2^{-\frac{1}{2}})$.

(4) [**Marked**] Consider the following set:

$$S = \left\{ (x, y, z) \in \mathbb{R}^3 \mid \left(3 + y^2\right) \left(x^2 + z^2\right) = 1 \right\}.$$

(a) Show that S is a surface.

(b) Sketch S.

(c) Give a parametrisation of S such that $\left(-\frac{1}{2\sqrt{2}}, 1, \frac{1}{2\sqrt{2}}\right)$ lies in the image of S.

(d) Compute the tangent plane to S at $\left(-\frac{1}{2\sqrt{2}}, 1, \frac{1}{2\sqrt{2}}\right)$.

(a) Notice that S can be written as a level set,

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid h(x, y, z) = 1\},$$

where h is the function

$$h : \mathbb{R}^3 \to \mathbb{R}, \qquad h(x, y, z) = \left(3 + y^2\right)\left(x^2 + y^2\right).$$

The gradient of h then satisfies

$$\nabla h(x, y, z) = \left(2x\left(3 + y^2\right), 2y\left(x^2 + z^2\right), 2z\left(3 + y^2\right)\right)_{(x,y,z)}.$$

Thus, $\nabla h(x, y, z)$ vanishes if and only if

$$x\left(3 + y^2\right) = 0, \qquad y\left(x^2 + z^2\right) = 0, \qquad z\left(3 + y^2\right) = 0.$$

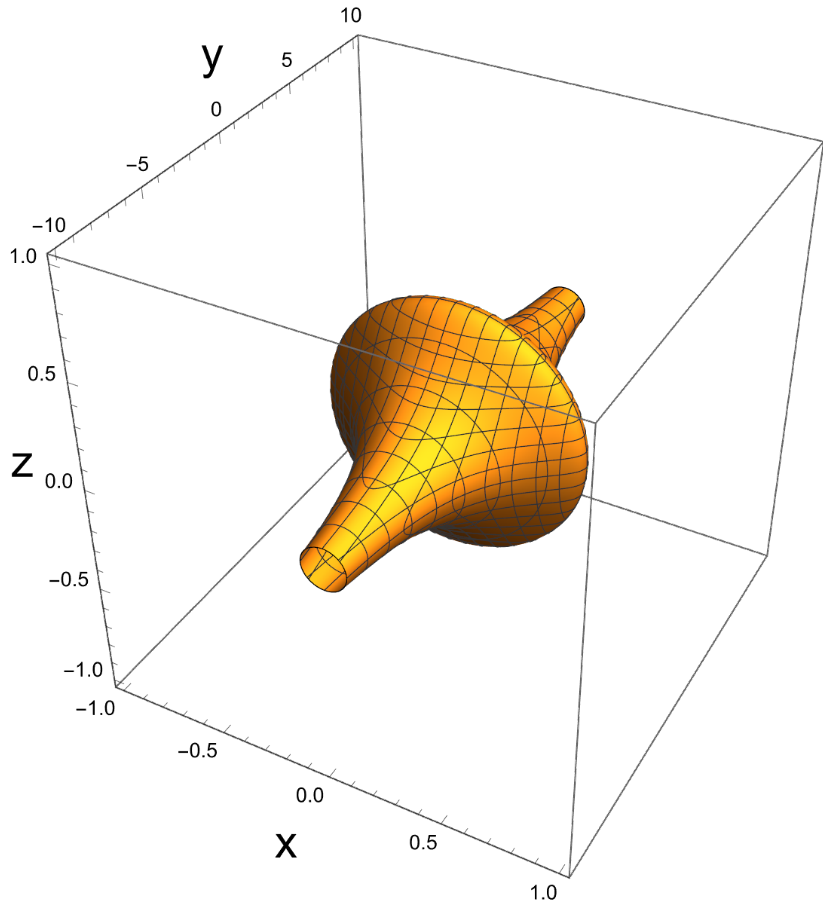
In particular, $\nabla h(x, y, z)$ vanishes if and only if $x = z = 0$ (while y is allowed to take any value). [1 mark for mostly correct reasoning to this point]

Since for any point $(0, y, 0)$ at which ∇h vanishes, we have

$$h(0, y, 0) = 0 \neq 1,$$

it hence follows that $\nabla h(x, y, z)$ does not vanish at any $(x, y, z) \in S$. Consequently, the level theorem implies S is indeed a surface. [1 mark for mostly correct reasoning]

(b) A drawing of S is given here:



[2 marks for mostly correct drawing]

(c) The easiest way to parametrise S is to set $y = v$ and $x = \frac{\cos u}{\sqrt{3+v^2}}$ $z = \frac{\sin u}{\sqrt{3+v^2}}$.
As a result, our parametrisation will be given by the formula

$$\sigma(u, v) = \left(\frac{\cos u}{\sqrt{3+v^2}},\ v,\ \frac{\sin u}{\sqrt{3+v^2}} \right).$$

It remains to determine a viable domain. If we wish to be injective, we must pick $0 < u < 2\pi$, whereas v is unrestricted. This maps onto all of S minus a line. We can instead choose a non-injective map which covers the whole surface. Both are fine.

Combining the above yields the parametrisation

$$\sigma : \{(u, v) \in \mathbb{R}^2\} \to S, \qquad \sigma(u, v) = \left(\frac{\cos u}{\sqrt{3+v^2}},\ v,\ \frac{\sin u}{\sqrt{3+v^2}} \right).$$

We can now verify that $\left(-\frac{1}{2\sqrt{2}}, 1, \frac{1}{2\sqrt{2}}\right) = \sigma\left(\frac{3\pi}{4}, 1\right)$.
[1 mark for correct parametrisation] [1 mark for correct domain]

(c) We use the parametrisation σ from part (b). First, note that

$$\partial_1\sigma(u,v) = \left(\frac{-\sin u}{\sqrt{3+v^2}}, 0, \frac{\cos u}{\sqrt{3+v^2}}\right), \qquad \partial_2\sigma(u,v) = \left(\frac{-v\cos u}{(3+v^2)^{3/2}}, 1, \frac{-v\sin u}{(3+v^2)^{3/2}}\right).$$

Since $\left(-\frac{1}{2\sqrt{2}}, 1, \frac{1}{2\sqrt{2}}\right) = \sigma\left(\frac{3\pi}{4}, 1\right)$, we must evaluate the above at $(u,v) = \left(\frac{3\pi}{4}, 1\right)$:

$$\partial_1\sigma\left(\frac{3\pi}{4}, 1\right) = \left(-\frac{1}{2\sqrt{2}}, 0, -\frac{1}{2\sqrt{2}}\right), \qquad \partial_2\sigma\left(\frac{3\pi}{4}, 1\right) = \left(\frac{1}{8\sqrt{2}}, 1, -\frac{1}{8\sqrt{2}}\right).$$

Therefore, by definition, the tangent plane to S at $\left(-\frac{1}{2\sqrt{2}}, 1, \frac{1}{2\sqrt{2}}\right)$ is

$$T_{\left(-\frac{1}{2\sqrt{2}},1,\frac{1}{2\sqrt{2}}\right)}S = T_\sigma\left(\frac{3\pi}{4}, 1\right) = \left\{ a\cdot\left(-\frac{1}{2\sqrt{2}}, 0, -\frac{1}{2\sqrt{2}}\right)_{\left(-\frac{1}{2\sqrt{2}},1,\frac{1}{2\sqrt{2}}\right)} + b\cdot\left(\frac{1}{8\sqrt{2}}, 1, -\frac{1}{8\sqrt{2}}\right)_{\left(-\frac{1}{2\sqrt{2}},1,\frac{1}{2\sqrt{2}}\right)} \,\middle|\, a, b \in \mathbb{R} \right\}.$$

[1 mark for almost correct answer]

(5) [Tutorial] Consider the *two-sheeted hyperboloid*:

$$\mathcal{H} = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = -1\}.$$

(a) Show that $\mathcal{H}$ is a surface.

(b) Give a sketch of $\mathcal{H}$.

(c) Give a parametrisation of $\mathcal{H}$ that passes through the point $(1, -1, \sqrt{3})$.

(d) Compute the tangent plane to $\mathcal{H}$ at the point $(1, -1, \sqrt{3})$.

(a) First, note that $\mathcal{H}$ can be written as a level set,

$$\mathcal{H} = \{(x,y,z) \in \mathbb{R}^3 \mid h(x,y,z) = -1\},$$

where $h : \mathbb{R}^3 \to \mathbb{R}$ is the function given by

$$h(x,y,z) = x^2 + y^2 - z^2.$$

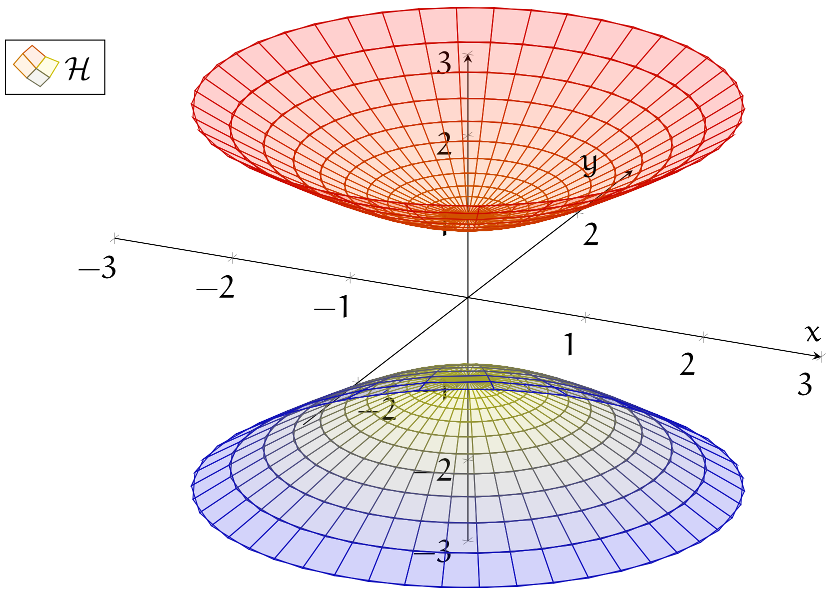
Note that the gradient of h satisfies

$$\nabla h(x, y, z) = (2x,\ 2y,\ -2z)_{(x,y,z)},$$

which vanishes only when $(x, y, z) = (0, 0, 0)$.

Since $(0, 0, 0) \notin \mathcal{H}$ (which follows since $h(0, 0, 0) = 0 \neq -1$), the level set theorem (see the lectures or the lecture notes) implies that the set $\mathcal{H}$ is indeed a surface.

(b) A sketch of $\mathcal{H}$ is provided below:



(c) Here, the most straightforward approach is to set our parameters u and v to be x and y, respectively. Then, the defining equation for $\mathcal{H}$ implies that $z^2 = 1 + u^2 + v^2$, hence

$$z = \pm\sqrt{1 + u^2 + v^2}.$$

Since we want the point $(1, -1, \sqrt{3})$ (which has positive z-value) to be in our parametrisation, we choose the "+" branch for z. This leads us to the following parametrisation of $\mathcal{H}$:

$$\sigma : \mathbb{R}^2 \to \mathcal{H}, \qquad \sigma(u, v) = \left(u,\ v,\ \sqrt{1 + u^2 + v^2}\right).$$

(Observe in particular that $(1, -1, \sqrt{3}) = \sigma(1, -1)$.)

For those of you who are sufficiently comfortable with the hyperbolic functions, you could also see that another correct parametrisation of $\mathcal{H}$ is given by

$$\tau : \mathbb{R}^2 \to \mathcal{H}, \qquad \tau(u, v) = (\cos u \sinh v,\ \sin u \sinh v,\ \cosh v).$$

(d) We compute the tangent plane using the parametrisation σ from (c). First, we have

$$\partial_1\sigma(u, v) = \left(1,\ 0,\ \frac{u}{\sqrt{1+u^2+v^2}}\right), \qquad \partial_2\sigma(u, v) = \left(0,\ 1,\ \frac{v}{\sqrt{1+u^2+v^2}}\right),$$
$$\partial_1\sigma(1, -1) = \left(1,\ 0,\ \frac{1}{\sqrt{3}}\right), \qquad \partial_2\sigma(1, -1) = \left(0,\ 1,\ -\frac{1}{\sqrt{3}}\right),$$

As a result, we conclude that

$$\begin{aligned} T_{(1,-1,\sqrt{3})}\mathcal{H} &= T_\sigma(1, -1) \\ &= \left\{ a \cdot \left(1,\ 0,\ \frac{1}{\sqrt{3}}\right)_{(1,-1,\sqrt{3})} + b \cdot \left(0,\ 1,\ -\frac{1}{\sqrt{3}}\right)_{(1,-1,\sqrt{3})} \,\middle|\, a, b \in \mathbb{R} \right\}. \end{aligned}$$

(6) *(Let's be self-sufficient)* For each of the following surfaces S and points $\mathbf{p} \in S$:

(i) Show that S is a surface.

(ii) Compute the tangent plane to S at $\mathbf{p}$.

(Unlike in Questions (4) and (5), you are not given a parametrisation of S. You will have to find your own in order to compute the tangent plane.)

(a) *Hyperbolic paraboloid:*

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x = yz\}, \qquad \mathbf{p} = (-6, 2, -3).$$

(b) *Cylinder:*

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + z^2 = 9\}, \qquad \mathbf{p} = \left(-\frac{3}{\sqrt{2}},\ 7,\ \frac{3}{\sqrt{2}}\right).$$

(a) (i) Note S can be written as a level set,

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid h(x, y, z) = 0\},$$

where h is the function

$$h : \mathbb{R}^3 \to \mathbb{R}, \qquad h(x, y, z) = x - yz.$$

Moreover, the gradient of h satisfies

$$\nabla h(x, y, z) = (1, -z, -y)_{(x,y,z)}, \qquad (x, y, z) \in \mathbb{R}^3,$$

which never vanishes. Thus, S is a surface by the level set theorem.

(ii) S is most easily parametrised by taking $y = u$ and $z = v$:

$$\sigma : \mathbb{R}^2 \to S, \qquad \sigma(u, v) = (uv,\ u,\ v).$$

Note in particular that $(-6, 2, -3) = \sigma(2, -3)$.

To compute the tangent plane, we first calculate

$$\partial_1\sigma(u, v) = (v, 1, 0), \qquad \partial_2\sigma(u, v) = (u, 0, 1),$$
$$\partial_1\sigma(2, -3) = (-3, 1, 0), \qquad \partial_2\sigma(2, -3) = (2, 0, 1).$$

Thus, by the definition of the tangent plane, we conclude that

$$T_{(-6,2,-3)}S = T_\sigma(2, -3) = \left\{ a \cdot (-3, 1, 0)_{(-6,2,-3)} + b \cdot (2, 0, 1)_{(-6,2,-3)} \,\middle|\, a, b \in \mathbb{R} \right\}.$$

(b) (i) First, observe that S can be written as a level set,

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid g(x, y, z) = 9\},$$

where g is the function

$$g : \mathbb{R}^3 \to \mathbb{R}, \qquad g(x, y, z) = x^2 + z^2.$$

The gradient of g satisfies

$$\nabla g(x, y, z) = (2x,\ 0,\ 2z)_{(x,y,z)}, \qquad (x, y, z) \in \mathbb{R}^3,$$

which vanishes only when $x = z = 0$.

However, any point $(x, y, z) \in \mathbb{R}^3$ satisfying $x = z = 0$ cannot lie on S, since

$$x^2 + z^2 = 0 \neq 9.$$

Thus, $\nabla g(\mathbf{p})$ does not vanish for any $\mathbf{p} \in S$, and it follows that S is a surface.

(ii) We can parametrise S using the unusual cylindrical coordinates (note, however, that the cylinder now has radius 3 and is centred about the y-axis):

$$\sigma : \mathbb{R}^2 \to S, \qquad \sigma(u, v) = (3\cos u,\ v,\ 3\sin u).$$

Note in particular that $\mathbf{p} = \sigma(\frac{3\pi}{4}, 7)$.

Taking derivatives of σ then yields

$$\begin{aligned} \partial_1\sigma(u, v) &= (-3\sin u,\ 0,\ 3\cos u), & \partial_2\sigma(u, v) &= (0, 1, 0), \\ \partial_1\sigma\left(\frac{3\pi}{4}, 7\right) &= \left(-\frac{3}{\sqrt{2}},\ 0,\ -\frac{3}{\sqrt{2}}\right), & \partial_2\sigma\left(\frac{3\pi}{4}, 7\right) &= (0, 1, 0). \end{aligned}$$

As a result, we conclude that

$$\begin{aligned} T_{\left(-\frac{3}{\sqrt{2}}, 7, \frac{3}{\sqrt{2}}\right)} S &= T_\sigma\left(\frac{3\pi}{4}, 7\right) \\ &= \left\{ a \cdot \left(-\frac{3}{\sqrt{2}},\ 0,\ -\frac{3}{\sqrt{2}}\right)_{\left(-\frac{3}{\sqrt{2}}, 7, \frac{3}{\sqrt{2}}\right)} + b \cdot (0, 1, 0)_{\left(-\frac{3}{\sqrt{2}}, 7, \frac{3}{\sqrt{2}}\right)} \;\middle|\; a, b \in \mathbb{R} \right\}. \end{aligned}$$

(7) *(Surfaces of revolution)* Let $f : (a, b) \to \mathbb{R}$ be a smooth function satisfying $f(x) > 0$ for every $x \in (a, b)$. From f, we can then define the set

$$\mathcal{R} = \{(x, y, z) \in \mathbb{R}^3 \mid a < x < b,\ y^2 + z^2 = [f(x)]^2\}.$$

In particular, $\mathcal{R}$ is the *surface of revolution* obtained by taking the graph of f (in the xy-plane) and rotating it (in 3-dimensional space) around the x-axis.

(a) Show that $\mathcal{R}$ is indeed a surface.

(b) Give a parametrisation of $\mathcal{R}$ whose image is all of $\mathcal{R}$.

(c) Compute the tangent plane to $\mathcal{R}$ at the point $(x, 0, f(x))$, for any $x \in (a, b)$.

(a) Notice that $\mathcal{R}$ can be written as a level set,

$$\mathcal{R} = \{(x, y, z) \in (a, b) \times \mathbb{R}^2 \mid k(x, y, z) = 0\},$$

where k is the (smooth) function

$$k : (a, b) \times \mathbb{R}^2 \to \mathbb{R}, \qquad k(x, y, z) = y^2 + z^2 - [f(x)]^2.$$

Moreover, its gradient satisfies,

$$\nabla k(x, y, z) = (-2f(x)f'(x), 2y, 2z)_{(x,y,z)}, \qquad (x, y, z) \in (a, b) \times \mathbb{R}^2.$$

Note $\nabla k(x, y, z)$ could only possibly vanish when $y = z = 0$. But, such a point cannot be on $\mathcal{R}$, since $y^2 + z^2 = 0 < [f(x)]^2$. Thus, by the level set theorem, $\mathcal{R}$ must be a surface.

(b) One straightforward parametrisation is the following:

$$\sigma : (a, b) \times \mathbb{R} \to \mathcal{R}, \qquad \sigma(u, v) = (u,\ f(u) \cos v,\ f(u) \sin v).$$

(Intuitively, at each x-value u, the points of $\mathcal{R}$ form a circle in the yz-plane of radius $f(u)$.)

(c) First, note that for any $x \in (a, b)$, we have that

$$(x, 0, f(x)) = \sigma\left(x, \frac{\pi}{2}\right).$$

Furthermore, we directly compute

$$\partial_1\sigma(u, v) = (1,\ f'(u) \cos v,\ f'(u) \sin v), \qquad \partial_2\sigma(u, v) = (0,\ -f(u) \sin v,\ f(u) \cos v),$$
$$\partial_1\sigma\left(x, \frac{\pi}{2}\right) = (1,\ 0,\ f'(x)), \qquad \partial_2\sigma\left(x, \frac{\pi}{2}\right) = (0,\ -f(x),\ 0).$$

As a result, the tangent plane to $\mathcal{R}$ at $(x, 0, f(x))$ is

$$\begin{aligned} T_{(x,0,f(x))}\mathcal{R} &= T_\sigma\left(x, \frac{\pi}{2}\right) \\ &= \left\{a \cdot (1,\ 0,\ f'(x))_{(x,\,0,\,f(x))} + b \cdot (0,\ -f(x),\ 0)_{(x,\,0,\,f(x))} \,\middle|\, a, b \in \mathbb{R}\right\} \\ &= \left\{a \cdot (1,\ 0,\ f'(x))_{(x,\,0,\,f(x))} + b \cdot (0,\ 1,\ 0)_{(x,\,0,\,f(x))} \,\middle|\, a, b \in \mathbb{R}\right\}. \end{aligned}$$

(8) *(Fun with stereographic projections)* Consider the parametric surface

$$\sigma : \mathbb{R}^2 \to \mathbb{R}^3, \qquad \sigma(u, v) = \mathbf{p},$$

where $\mathbf{p}$ is the (unique) point of $\mathbb{S}^2 \setminus \{(0, 0, 1)\}$ that lies on the line through the points $(u, v, 0)$ and $(0, 0, 1)$. (The function σ is called the *inverse stereographic projection.*)

(a) Show that σ can be described by the formula

$$\sigma(u, v) = \left(\frac{2u}{1 + u^2 + v^2}, \frac{2v}{1 + u^2 + v^2}, \frac{-1 + u^2 + v^2}{1 + u^2 + v^2} \right), \qquad (u, v) \in \mathbb{R}^2.$$

(b) Show that σ is both injective and regular.

(c) Show that the image of σ is precisely $\mathbb{S}^2 \setminus \{(0, 0, 1)\}$.

(d) Use your knowledge of σ to construct the sphere $\mathbb{S}^2$ using only two regular and injective parametric surfaces.

(a) Fix any $(u, v) \in \mathbb{R}^2$. The line through $(0, 0, 1)$ and $(u, v, 0)$ can be parametrised as

$$\ell : \mathbb{R} \to \mathbb{R}^3, \qquad \ell(t) = (0, 0, 1) + [(u, v, 0) - (0, 0, 1)]t = (ut, \; vt, \; 1 - t).$$

Thus, we need to find the point $\ell(t)$ which lies on $\mathbb{S}^2 \setminus \{(0, 0, 1)\}$. For this, we solve

$$1 = |\ell(t)|^2 = t^2(1 + u^2 + v^2) + 1 - 2t$$

for t. Note the above has the solutions $t = 0$ and $t = 2(1+u^2+v^2)^{-1}$. The former corresponds to $\ell(0) = (0, 0, 1)$, while the latter corresponds to our desired point on $\mathbb{S}^2 \setminus \{(0, 0, 1)\}$:

$$\begin{aligned} \sigma(u, v) &= \ell\left(\frac{2}{1 + u^2 + v^2} \right) \\ &= \left(\frac{2u}{1 + u^2 + v^2}, \frac{2v}{1 + u^2 + v^2}, 1 - \frac{2}{1 + u^2 + v^2} \right) \\ &= \left(\frac{2u}{1 + u^2 + v^2}, \frac{2v}{1 + u^2 + v^2}, \frac{-1 + u^2 + v^2}{1 + u^2 + v^2} \right). \end{aligned}$$

(b) Suppose $\sigma(u_1, v_1) = \sigma(u_2, v_2)$. Then, by the definition of σ:

$$\frac{2u_1}{1 + u_1^2 + v_1^2} = \frac{2u_2}{1 + u_2^2 + v_2^2}, \tag{1}$$

$$\frac{2v_1}{1+u_1^2+v_1^2} = \frac{2v_2}{1+u_2^2+v_2^2}, \tag{2}$$

$$\frac{-1+u_1^2+v_1^2}{1+u_1^2+v_1^2} = \frac{-1+u_2^2+v_2^2}{1+u_2^2+v_2^2}. \tag{3}$$

Note that (3) can be rewritten as

$$1 - \frac{2}{1+u_1^2+v_1^2} = 1 - \frac{2}{1+u_2^2+v_2^2},$$

from which we conclude that $1+u_1^2+v_1^2 = 1+u_2^2+v_2^2$. Applying this to (1) and (2) yields $u_1 = u_2$ and $v_1 = v_2$, respectively. As a result, σ is indeed injective.

Next, we compute the partial derivatives of σ:

$$\partial_1\sigma(u,v) = \frac{2}{(1+u^2+v^2)^2}(1-u^2+v^2,\ -2uv,\ 2u),$$
$$\partial_2\sigma(u,v) = \frac{2}{(1+u^2+v^2)^2}(-2uv,\ 1+u^2-v^2,\ 2v).$$

Taking a cross product, we see that

$$|\partial_1\sigma(u,v) \times \partial_2\sigma(u,v)| = \frac{4}{(1+u^2+v^2)^4}|(1+u^2+v^2)(-2u,\ -2v,\ 1-u^2-v^2)|$$
$$= \frac{4}{(1+u^2+v^2)^2}|\sigma(u,v)|.$$

Since $\sigma(u,v) \in \mathbb{S}^2$, we have that $|\sigma(u,v)| = 1$, and hence

$$|\partial_1\sigma(u,v) \times \partial_2\sigma(u,v)| = \frac{4}{(1+u^2+v^2)^2} \neq 0, \qquad (u,v) \in \mathbb{R}^2.$$

As a result, σ is regular.

(c) We already know $\sigma(u,v) \in \mathbb{S}^2 \setminus \{(0,0,1)\}$ for any $(u,v) \in \mathbb{R}^2$. Thus, it remains to show that given any $(x,y,z) \in \mathbb{S}^2 \setminus \{(0,0,1)\}$, there is some $(u,v) \in \mathbb{R}^2$ such that $(x,y,z) = \sigma(u,v)$.

By the definition of σ, our desired (u,v) must satisfy that $(u,v,0)$ lies on the line L through (x,y,z) and $(0,0,1)$. Observe that L can be parametrised as

$$L : \mathbb{R} \to \mathbb{R}^3, \qquad L(t) = (0,0,1) + [(x,y,z) - (0,0,1)]t = (xt, yt, 1+(z-1)t).$$

Noting that

$$L\left(\frac{1}{1-z}\right) = \left(\frac{x}{1-z}, \frac{y}{1-z}, 0\right),$$

we can then guess that

$$(u, v) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right).$$

Finally, to check that $\sigma(u, v)$ indeed equals (x, y, z), we can directly check

$$\sigma\left(\frac{x}{1-z}, \frac{y}{1-z}\right) = \left(\frac{2(1-z)x}{(1-z)^2 + x^2 + y^2}, \frac{2(1-z)y}{(1-z)^2 + x^2 + y^2}, 1 - \frac{2(1-z)^2}{(1-z)^2 + x^2 + y^2}\right).$$

Noting that $x^2 + y^2 + z^2 = 1$ (since $(x, y, z) \in \mathbb{S}^2$), the above simplifies to

$$\sigma\left(\frac{x}{1-z}, \frac{y}{1-z}\right) = \left(\frac{2(1-z)x}{2-2z}, \frac{2(1-z)y}{2-2z}, 1 - \frac{2(1-z)^2}{2-2z}\right) = (x, y, z).$$

Consequently, we conclude that the image of σ is precisely $\mathbb{S}^2 \setminus \{(0, 0, 1)\}$.

(d) From parts (a)–(c), we see that σ is an injective parametrisation of $\mathbb{S}^2$, and that the image of σ is $\mathbb{S}^2 \setminus \{(0, 0, 1)\}$. Define in addition the parametric surface $\tau : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$\tau(u, v) = \left(\frac{2u}{1 + u^2 + v^2}, \frac{2v}{1 + u^2 + v^2}, \frac{1 - u^2 - v^2}{1 + u^2 + v^2}\right).$$

Since τ is simply σ with the z-component negated, it follows that τ is an injective parametrisation of $\mathbb{S}^2$, and its image is $\mathbb{S}^2 \setminus \{(0, 0, -1)\}$. Thus, σ and τ together cover all of $\mathbb{S}^2$.