MTH5113 (2023/24): Problem Sheet 6
Solutions

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(1) *(Warm-up)* For each of the sets C and points $\mathbf{p} \in C$ given below:

(i) Show that C is a curve.

(ii) Sketch C, and indicate the point $\mathbf{p}$ on C.

(iii) Give a parametrisation of C that passes through $\mathbf{p}$.

(a) *Hyperbola:*

$$C = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 = -1\}, \qquad \mathbf{p} = (0, -1).$$

(b) *Cubic:*

$$C = \{(x, y) \in \mathbb{R}^2 \mid (x - 2)^3 = y - 3\}, \qquad \mathbf{p} = (0, -5).$$

(c) *Ellipse:*

$$C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + 6x + 4y^2 - 8y = 3\}, \qquad \mathbf{p} = (-3, -1).$$

(a) (i) Note that C can be written as the following level set:

$$C = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = -1\},$$

where f is the function

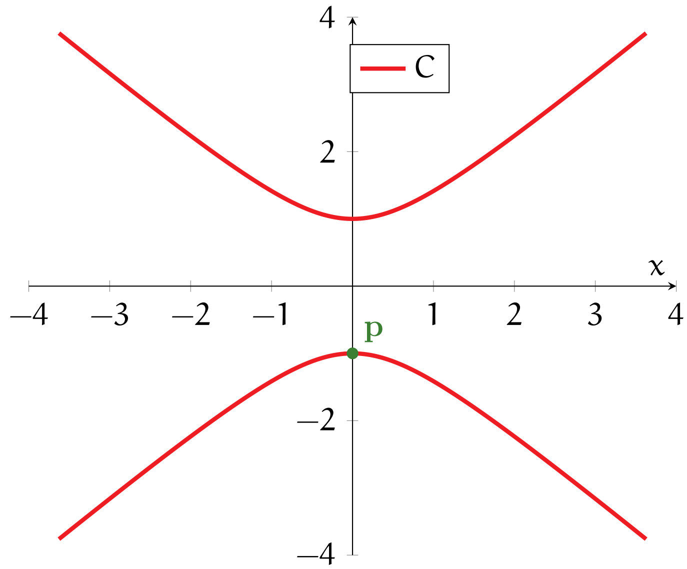
$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad f(x, y) = x^2 - y^2.$$

Observe that the gradient of f satisfies

$$\nabla f(x, y) = (2x, -2y)_{(x,y)},$$

which vanishes only when $(x, y) = (0, 0) \notin C$. Thus, the "level set theorem" (from either the lectures or the lecture notes) implies that C is indeed a curve.

(ii) A sketch is given below, with C in red and $\mathbf{p}$ in green:



(iii) One approach is to set x to be the parameter t. Note that y must satisfy

$$y^2 = 1 + x^2, \qquad y = \pm\sqrt{1 + x^2} = \pm\sqrt{1 + t^2}.$$

Since this parametrisation must pass through $\mathbf{p} = (0, -1)$, it follows that we must choose the negative branch in the above. As a result, one possible parametrisation of C is

$$\gamma : \mathbb{R} \to \mathbb{R}^2, \qquad \gamma(t) = \left(t, -\sqrt{1 + t^2}\right).$$

(Note in particular γ maps out the bottom half of C, and that $\gamma(0) = (0, -1)$.)

If you like hyperbolic functions, then a slicker parametrisation of the bottom half of C is

$$\lambda : \mathbb{R} \to \mathbb{R}^2, \qquad \lambda(t) = (\sinh t, -\cosh t).$$

(Note in particular that $\lambda(0) = (0, -1)$.)

(b) (i) Here, we need simply note that C is the graph of the function

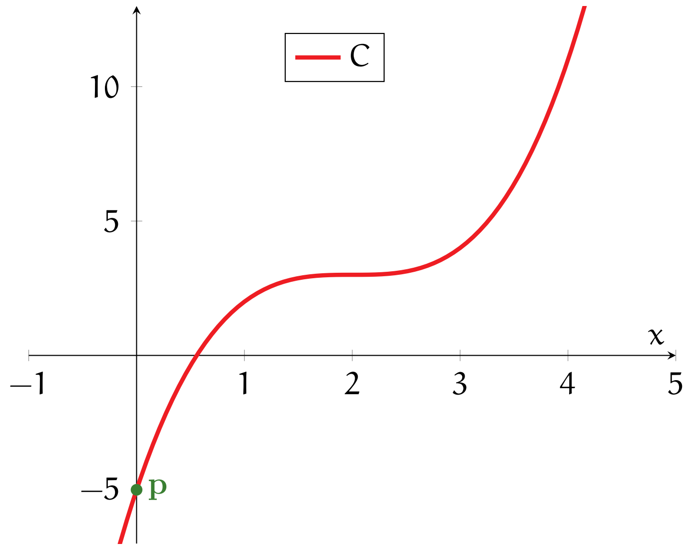
$$c : \mathbb{R} \to \mathbb{R}, \qquad c(x) = (x - 2)^3 + 3,$$

that is,

$$C = \{(t, c(t)) \mid t \in \mathbb{R}\}.$$

It follows (from theorems in lectures or the lecture notes) that C is indeed a curve.

(ii) A sketch is given below, with C in red and **p** in green:



(iii) Since C is the graph of c from part (i), we have a natural way to parametrise all of C:

$$\gamma_c : \mathbb{R} \to \mathbb{R}^2, \qquad \gamma_c(t) = (t,\ c(t)) = (t,\ (t-2)^3 + 3).$$

Note in particular that $\gamma_c(0) = (0,\ -5)$.

(c) This one is a bit trickier. First, we can write the defining equation for C as

$$16 = (x^2 + 6x + 9) + 4(y^2 - 2y + 4) = (x+3)^2 + 4(y-1)^2.$$

In other words, we can express

$$C = \{(x,y) \in \mathbb{R}^2 \mid (x+3)^2 + 4(y-1)^2 = 16\},$$

from which we can conclude that C is an ellipse centred at $(-3, 1)$.

(i) To show C is a curve, we first note that it is a level set of the function

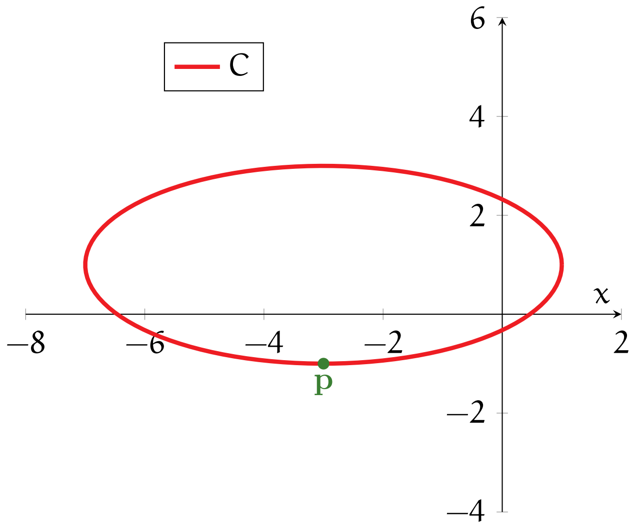
$$g : \mathbb{R}^2 \to \mathbb{R}, \qquad g(x,y) = (x+3)^2 + 4(y-1)^2.$$

Note that the gradient of g satisfies

$$\nabla g(x,y) = (2(x+3),\ 8(y-1))_{(x,y)},$$

which only vanishes when $(x, y) = (-3, 1)$. Since $(-3, 1) \notin C$ (this can be checked using the definition of C), it follows that C is indeed a curve.

(ii) A sketch is given below, with C in red and $\mathbf{p}$ in green:



(iii) The nicest parametrisation is obtained by relating C to a circle. Setting

$$\tilde{x} = x + 3, \qquad \tilde{y} = 2(y - 1),$$

then the defining equation for C is becomes

$$\tilde{x}^2 + \tilde{y}^2 = 4^2.$$

Since the above describes a circle about the origin of radius 4, this can be parametrised as $\tilde{x} = 4\cos t$ and $\tilde{y} = 4\sin t$. Putting the above in terms of x and y yields

$$x = -3 + 4\cos t, \qquad y = 1 + 2\sin t.$$

The above implies that the following is a valid parametrisation of (all of) C:

$$\gamma : \mathbb{R} \to \mathbb{R}^2, \qquad \gamma(t) = (-3 + 4\cos t,\ 1 + 2\sin t).$$

(Note, in particular, that $(-3, -1) = \gamma(-\frac{\pi}{2})$.)

Alternatively, one can set $x = t$ and solve for y. This yields

$$\lambda : (-7, 1) \to \mathbb{R}^2, \qquad \lambda(t) = \left(t,\ 1 - \sqrt{4 - \frac{1}{4}(t+3)^2} \right).$$

In this case, $(-3, -1) = \lambda(-3)$.

(2) *(Fun with plotting)* The following are exercises involving sketching parametric surfaces. Do make use of computer programs or webpages (see the links in the *Additional Resources* section on the *QMPlus page*) to help you with your sketches.

(a) *Sphere:* Consider the following parametric surface:

$$\sigma : \mathbb{R}^2 \to \mathbb{R}^3, \qquad \sigma(u, v) = (\cos u \sin v,\ \sin u \sin v,\ \cos v).$$

(i) Sketch the paths obtained from σ by holding v constant, with values

$$v_0 = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi.$$

(ii) Sketch the paths obtained from σ by holding u constant, with values

$$u_0 = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi.$$

(iii) Sketch the image of σ.

(b) *Hyperboloid:* Consider the following parametric surface:

$$\mathbf{h} : \mathbb{R}^2 \to \mathbb{R}^3, \qquad \mathbf{h}(u, v) = (\cos u \cosh v,\ \sin u \cosh v,\ \sinh v).$$

(i) Sketch the paths obtained from $\mathbf{h}$ by holding v constant, with values

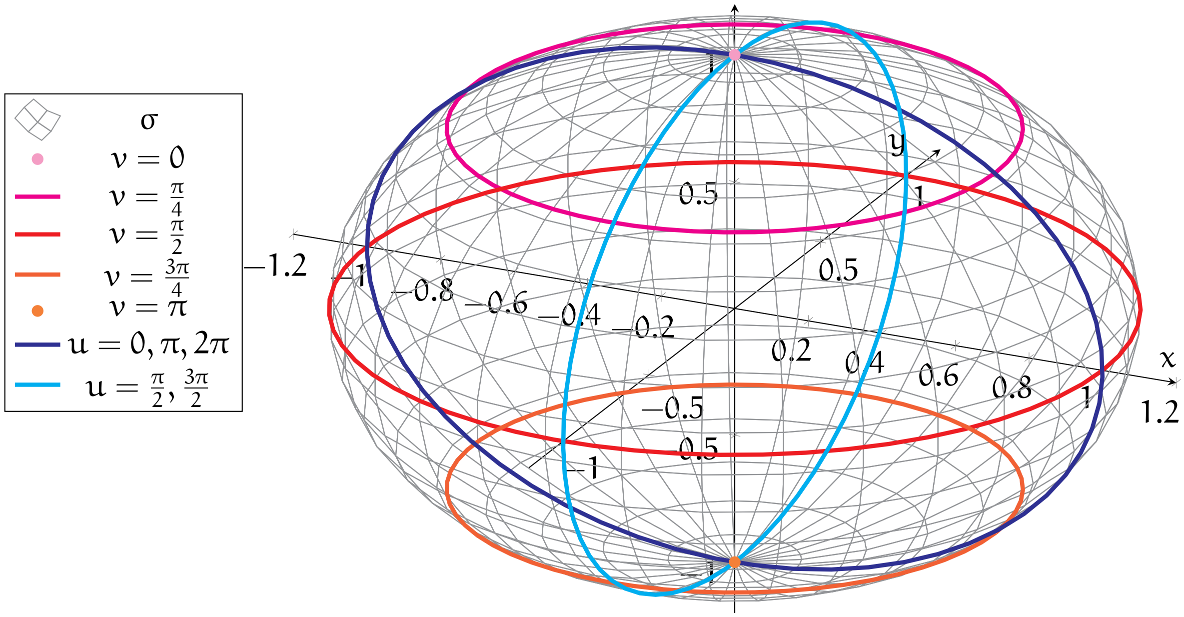
$$v_0 = -2, -1, 0, 1, 2.$$

(ii) Sketch the paths obtained from $\mathbf{h}$ by holding u constant, with values

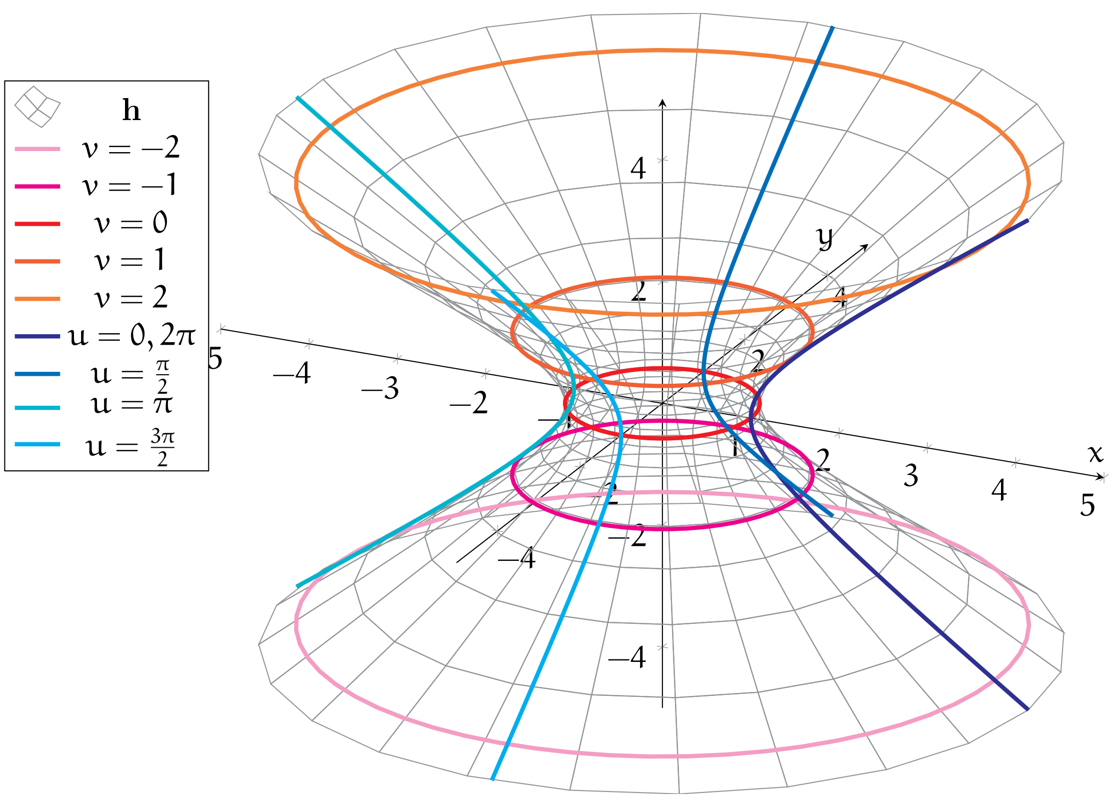
$$u_0 = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi.$$

(iii) Sketch the image of $\mathbf{h}$.

(a) A sketch of σ is given below; moreover, the paths obtained by holding v constant are drawn in red, while the paths obtained by holding u constant are drawn in blue:



(b) A sketch of $\mathbf{h}$ is given below; moreover, the paths obtained by holding v constant are drawn in red, while the paths obtained by holding u constant are drawn in blue:



(3) *(Warm-up)* For each of the following parametric surfaces σ and parameters (u_0, v_0), compute the tangent plane to σ at (u_0, v_0):

(a) σ is the parametric *cylinder*,

$$\sigma : \mathbb{R}^2 \to \mathbb{R}^3, \qquad \sigma(u, v) = (\cos u,\ \sin u,\ v),$$

and $(u_0, v_0) = (\frac{\pi}{2}, -1)$.

(b) σ is the parametric *one-sheeted hyperboloid*,

$$\sigma : \mathbb{R}^2 \to \mathbb{R}^3, \qquad \sigma(u, v) = (\cos u \cosh v,\ \sin u \cosh v,\ \sinh v).$$

and $(u_0, v_0) = (\pi, 1)$.

(a) We begin by differentiating σ:

$$\partial_1 \sigma(u, v) = (-\sin u,\ \cos u,\ 0), \qquad \partial_2 \sigma(u, v) = (0,\ 0,\ 1).$$

Evaluating the above at $(\frac{\pi}{2}, -1)$ yields

$$\sigma\left(\frac{\pi}{2}, -1\right) = (0, 1, -1), \qquad \partial_1 \sigma\left(\frac{\pi}{2}, -1\right) = (-1, 0, 0), \qquad \partial_2 \sigma\left(\frac{\pi}{2}, -1\right) = (0, 0, 1).$$

Thus, by definition, we conclude that

$$\begin{aligned} T_\sigma\left(-\frac{\pi}{2}, -1\right) &= \left\{ a \cdot \partial_1 \sigma\left(\frac{\pi}{2}, -1\right)_{\sigma(\frac{\pi}{2}, -1)} + b \cdot \partial_2 \sigma\left(\frac{\pi}{2}, -1\right)_{\sigma(\frac{\pi}{2}, -1)} \,\middle|\, a, b \in \mathbb{R} \right\} \\ &= \left\{ a \cdot (-1, 0, 0)_{(0,1,-1)} + b \cdot (0, 0, 1)_{(0,1,-1)} \,\middle|\, a, b \in \mathbb{R} \right\}. \end{aligned}$$

(b) Taking partial derivatives of σ yields

$$\begin{aligned} \partial_1 \sigma(u, v) &= (-\sin u \cosh v,\ \cos u \cosh v,\ 0), \\ \partial_2 \sigma(u, v) &= (\cos u \sinh v,\ \sin u \sinh v,\ \cosh v). \end{aligned}$$

Evaluating the above at $(\pi, 1)$, we obtain

$$\begin{aligned} \sigma(\pi, 1) &= (-\cosh 1,\ 0,\ \sinh 1), \\ \partial_1 \sigma(\pi, 1) &= (0,\ -\cosh 1,\ 0), \end{aligned}$$

$$\partial_2 \sigma(\pi, 1) = (-\sinh 1,\ 0,\ \cosh 1).$$

Thus, by definition,

$$T_\sigma(\pi, 1) = \left\{ a\,(0, -\cosh 1, 0)_{(-\cosh 1, 0, \sinh 1)} + b\,(-\sinh 1, 0, \cosh 1)_{(-\cosh 1, 0, \sinh 1)} \,\middle|\, a, b \in \mathbb{R} \right\}.$$

The above answer can be further expanded by recalling the formulas

$$\cosh 1 = \frac{1}{2}(e + e^{-1}), \qquad \sinh 1 = \frac{1}{2}(e - e^{-1}).$$

(4) *(Introduction to curve integrals)* One can also define an intermediate notion of curve integration of vector fields over *parametric curves*. More specifically:

> **Definition.** Let $\gamma : (a, b) \to \mathbb{R}^n$ be a parametric curve, and let $\mathbf{F}$ be a vector field that is defined on the image of γ. We then define the *curve integral* of $\mathbf{F}$ over γ by
>
> $$\int_\gamma \mathbf{F} \cdot ds = \int_a^b [\mathbf{F}(\gamma(t)) \cdot \gamma'(t)_{\gamma(t)}]\, dt.$$

For each of the following γ and $\mathbf{F}$, compute the curve integral of $\mathbf{F}$ over γ:

(a) γ is the regular parametric curve

$$\gamma : (0, 1) \to \mathbb{R}^3, \qquad \gamma(t) = (t,\ -t,\ 2t),$$

and $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ given by

$$\mathbf{F}(x, y, z) = (x, y, z)_{(x,y,z)}.$$

(b) γ is the regular parametric curve

$$\gamma : (0, 2\pi) \to \mathbb{R}^2, \qquad \gamma(t) = (\cos t,\ \sin t \cos t),$$

and $\mathbf{F}$ is the vector field on $\mathbb{R}^2$ given by

$$\mathbf{F}(x, y) = (x^2, 0)_{(x,y)}.$$

(a) The first step is to compute, for each $t \in (0, 1)$,

$$\gamma'(t) = (1, -1, 2), \qquad \mathbf{F}(\gamma(t)) = \mathbf{F}(t, -t, 2t) = (t, -t, 2t)_{(t,-t,2t)}.$$

Thus, by the above definition, we obtain

$$\begin{aligned}
\int_\gamma \mathbf{F} \cdot d\mathbf{s} &= \int_0^1 \left[\mathbf{F}(\gamma(t)) \cdot \gamma'(t)_{\gamma(t)}\right] dt \\
&= \int_0^1 \left[(t, -t, 2t)_{(t,-t,2t)} \cdot (1, -1, 2)_{(t,-t,2t)}\right] dt \\
&= \int_0^1 [(t, -t, 2t) \cdot (1, -1, 2)]\, dt \\
&= \int_0^1 6t\, dt \\
&= 3.
\end{aligned}$$

(b) Again, we begin by computing the necessary quantities:

$$\gamma'(t) = (-\sin t, \cos^2 t - \sin^2 t), \qquad F(\gamma(t)) = (\cos^2 t, 0)_{(\cos t, \sin t \cos t)}.$$

As a result, we see that

$$\begin{aligned}
\int_\gamma \mathbf{F} \cdot d\mathbf{s} &= \int_0^{2\pi} \left[\mathbf{F}(\gamma(t)) \cdot \gamma'(t)_{\gamma(t)}\right] dt \\
&= \int_0^{2\pi} [(\cos^2 t, 0) \cdot (-\sin t, \cos^2 t - \sin^2 t)]\, dt \\
&= -\int_0^{2\pi} \cos^2 t \sin t\, dt \\
&= \frac{1}{3}[\cos^3 t]_{t=0}^{t=2\pi} \\
&= 0.
\end{aligned}$$

(5) [Marked] *(Conservative forces)* Conservative forces have a very special property that we will explore in this problem. Consider a function $U : \mathbb{R}^2 \to \mathbb{R}$ which we will call *the potential*. The *conservative force* $\mathbf{F}$ associated to the potential U is given by

$$\mathbf{F} = -\nabla U\, .$$

(a) Consider the following potential

$$U(x, y) = -\frac{1}{\sqrt{x^2 + y^2 + 1}} \, .$$

Compute the force $\mathbf{F}$ associated to U and sketch the vector field associated to $\mathbf{F}$. (You should produce something like Figure 2.18 from the *lecture notes*.)

(b) Consider two curves C_1 and C_2 with injective parametrizations $\gamma_1 : (0, 2\pi) \to \mathbb{R}^2$ and $\gamma_2 : (0, 2\pi) \to \mathbb{R}^2$ given by

$$\gamma_1(t) = \left(0, \frac{t}{2\pi}\right) \, , \qquad \gamma_2(t) = \left(\sin(t), \frac{t}{2\pi}\right) \, ,$$

Sketch the curves C_1 and C_2.

(c) Compute the curve integrals $\int_{C_1} \mathbf{F} \cdot \mathbf{ds}$ and $\int_{C_2} \mathbf{F} \cdot \mathbf{ds}$. Do you notice anything?

(d) If you've done things correctly, you should have found $\int_{C_1} \mathbf{F} \cdot \mathbf{ds} = \int_{C_2} \mathbf{F} \cdot \mathbf{ds}$: *the answer is independent of the shape of the curve*! In fact, you may have noticed that

$$\int_{C_1} \mathbf{F} \cdot \mathbf{ds} = \int_{C_2} \mathbf{F} \cdot \mathbf{ds} = -\left(U(0, 1) - U(0, 0)\right) \, .$$

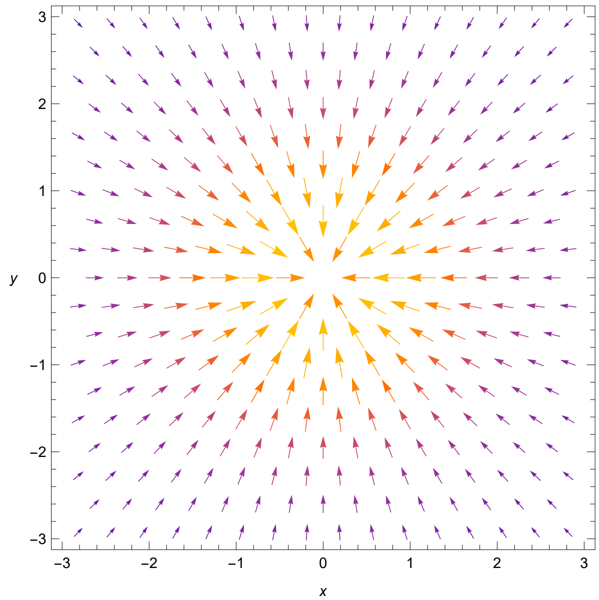
Does this remind you of anything? We will explore this further in section 5.4 of the lecture notes.

(a) To compute $\mathbf{F}$ need to compute the gradient of U:

$$\begin{aligned} \mathbf{F} &= -\nabla U = -\left(\partial_1 U, \partial_2 U\right)_{(x,y)} \\ &= \left(-\frac{x}{\left(x^2 + y^2 + 1\right)^{3/2}}, -\frac{y}{\left(x^2 + y^2 + 1\right)^{3/2}}\right)_{(x,y)} \end{aligned}$$

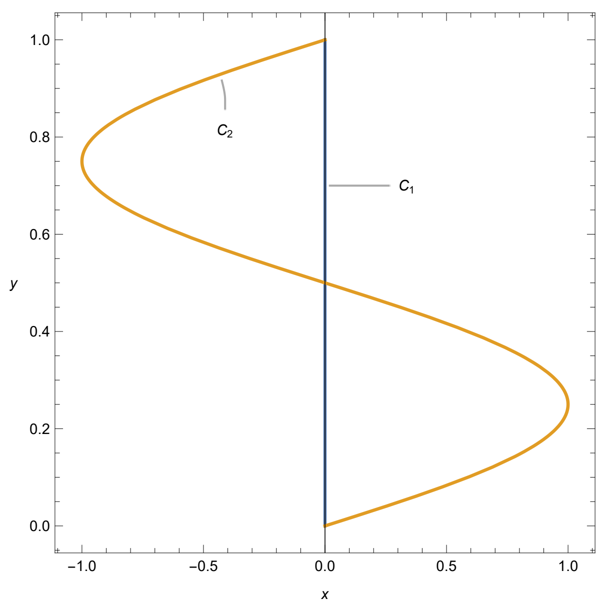
[1 mark for correct computation]

A sketch of this vector field is provided here



[1 mark for mostly correct sketch]

(b) A sketch of the two curves C_1 and C_2 is provided below



[1 mark for mostly correct sketch]

(c) Since parametrisations induce an orientation for a curve, we have two curves endowed with a natural orientation. [1 mark for correct orientation]

Let us first compute

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{s} = \int_0^{2\pi} \left[\mathbf{F}(\gamma_1(t))\cdot \gamma_1'(t)_{\gamma_1(t)}\right] dt$$

$$= \int_0^{2\pi} \left[\left(0, -\frac{t}{2\pi\left(1+\frac{t^2}{4\pi^2}\right)^{3/2}}\right)_{\left(0,\frac{t}{2\pi}\right)} \cdot \left(0, \frac{1}{2\pi}\right)_{\left(0,\frac{t}{2\pi}\right)} \right] dt$$

$$= \int_0^{2\pi} \left[-\frac{2\pi t}{(4\pi^2+t^2)^{3/2}} \right] dt$$

$$= \left.\frac{2\pi}{\sqrt{4\pi^2+t^2}}\right|_0^{2\pi}$$

$$= \frac{1}{\sqrt{2}} - 1$$

[1 mark for mostly correct manipulations. 1 mark for correct answer.]

Similarly for C_2 we have

$$\int_{C_2} \mathbf{F}\cdot d\mathbf{s} = \int_0^{2\pi} \left[\mathbf{F}(\gamma_2(t))\cdot \gamma_2'(t)_{\gamma_2(t)}\right] dt$$

$$= \int_0^{2\pi} \left[\left(\frac{-\sin(t)}{\left(1+\sin^2(t)+\frac{t^2}{4\pi^2}\right)^{3/2}}, \frac{-t}{2\pi\left(1+\sin^2(t)+\frac{t^2}{4\pi^2}\right)^{3/2}}\right)_{\left(\sin t,\frac{t}{2\pi}\right)} \cdot \left(\cos t, \frac{1}{2\pi}\right)_{\left(\sin t,\frac{t}{2\pi}\right)} \right] dt$$

$$= \int_0^{2\pi} \left[-\frac{2\pi\left(4\pi^2\sin t\cos t + t\right)}{\left(4\pi^2+4\pi^2\sin^2(t)+t^2\right)^{3/2}} \right] dt$$

$$= \left.\frac{2\pi}{\sqrt{4\pi^2+4\pi^2\sin^2(t)+t^2}}\right|_0^{2\pi}$$

$$= \frac{1}{\sqrt{2}} - 1$$

[1 mark for mostly correct manipulations. 1 mark for correct answer.]

(d) We have just shown that

$$-\int_{C_1} (\nabla U)\cdot d\mathbf{s} = -\int_{C_2} (\nabla U)\cdot d\mathbf{s} = -\left(U(0,1) - U(0,0)\right)\ ,$$

which is the difference between U and the start point and U at the end point of either curve. This should remind you of the *fundamental theorem of calculus.* [1 mark for any reasonable statement.]

(6) [**Tutorial**] For each of the following oriented curves C and vector fields $\mathbf{F}$:

(i) Give an *injective* parametrisation γ of C such that *the image of* γ *differs from* C *by only a finite number of points. Which orientation does* γ *generate?*

(ii) Compute the (curve) integral of $\mathbf{F}$ over the curve C.

(a) C is the *anticlockwise-oriented ellipse*,

$$C = \{(x,\ y) \in \mathbb{R}^2 \mid 3x^2 + 2y^2 = 6\},$$

and $\mathbf{F}$ is the vector field on $\mathbb{R}^2$ given by

$$\mathbf{F}(x, y) = (y,\ -x)_{(x,y)}.$$

(b) C is the *downward-oriented* (with decreasing z-value) *helical segment*,

$$C = \{(\cos t,\ \sin t,\ t) \mid t \in (0,\ 2\pi)\},$$

and $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ given by

$$\mathbf{F}(x, y, z) = (-y,\ x,\ 1)_{(x,y,z)}.$$

(a) (i) For this, we can borrow the formula from Question (5b) of Problem Sheet 4:

$$\gamma(t) = \left(\sqrt{2} \cdot \cos t,\ \sqrt{3} \cdot \sin t\right).$$

We can make it injective by restricting its domain so that it only traverses C once:

$$\gamma : (0, 2\pi) \to C, \qquad \gamma(t) = \left(\sqrt{2} \cdot \cos t,\ \sqrt{3} \cdot \sin t\right).$$

(In particular, the image of image is all of C except for its rightmost point $(\sqrt{2},\ 0)$.) Furthermore, notice that γ generates the anticlockwise orientation of C.

(ii) We can now integrate **F** using γ from part (i):

$$\int_C \mathbf{F} \cdot \mathbf{ds} = +\int_0^{2\pi} \left[\mathbf{F}(\gamma(t)) \cdot \gamma'(t)_{\gamma(t)}\right] dt.$$

By direct computations, we see that

$$\gamma'(t) = \left(-\sqrt{2} \cdot \sin t,\ \sqrt{3} \cdot \cos t\right),$$
$$\mathbf{F}(\gamma(t)) = \left(\sqrt{3} \cdot \sin t,\ -\sqrt{2} \cdot \cos t\right)_{\left(\sqrt{2}\cdot\cos t, \sqrt{3}\cdot\sin t\right)}.$$

Thus, combining the above yields

$$\begin{aligned}\int_C \mathbf{F} \cdot \mathbf{ds} &= \int_0^{2\pi} \left[\left(\sqrt{3} \cdot \sin t,\ -\sqrt{2} \cdot \cos t\right) \cdot \left(-\sqrt{2} \cdot \sin t,\ \sqrt{3} \cdot \cos t\right)\right] dt \\ &= -\sqrt{6} \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt \\ &= -2\sqrt{6}\,\pi.\end{aligned}$$

(b) (i) We can use the apparent parametrisation of C,

$$\lambda : (0,\ 2\pi) \to C, \qquad \lambda(t) = (\cos t,\ \sin t,\ t),$$

which is injective and whose image is all of C. Notice, however, that λ has the upward orientation, which is opposite to the given orientation of C.

(ii) The integral can now be computed using λ from part (i):

$$\begin{aligned}\int_C \mathbf{F} \cdot \mathbf{ds} &= -\int_0^{2\pi} \left[\mathbf{F}(\lambda(t)) \cdot \lambda'(t)_{\lambda(t)}\right] dt \\ &= -\int_0^{2\pi} [(-\sin t,\ \cos t,\ 1) \cdot (-\sin t,\ \cos t,\ 1)]\, dt \\ &= -\int_0^{2\pi} (\sin^2 t + \cos^2 t + 1)\, dt \\ &= -4\pi.\end{aligned}$$

(7) *(Exploring curvature)* Let $\gamma : I \to \mathbb{R}^n$ be any regular parametric curve. We then define the *curvature* of γ at $t \in I$ by the formula

$$\kappa_\gamma(t) = \frac{1}{|\gamma'(t)|} \left| \left(\frac{\gamma'}{|\gamma'|} \right)' (t) \right| .$$

(This can be viewed as the "change in the direction of γ per unit length"; see the *2019* version of the *MTH5113 lecture notes* for additional discussions of curvature.)

(a) Let $\mathbf{p}, \mathbf{v} \in \mathbb{R}^n$, and let ℓ be the parametric line,

$$\ell : \mathbb{R} \to \mathbb{R}^n, \qquad \ell(t) = \mathbf{p} + t\mathbf{v}.$$

Compute the curvature of ℓ at every $t \in \mathbb{R}$.

(b) Let $R > 0$, and let γ_R be the parametric circle of radius R:

$$\gamma_R : \mathbb{R} \to \mathbb{R}^2, \qquad \gamma_R(t) = (R\cos t,\, R\sin t).$$

Compute the curvature of γ_R at every $t \in \mathbb{R}$.

(c) Show that *curvature is independent of parametrisation.* More specificially, show that if γ is a reparametrisation of $\tilde{\gamma} : \tilde{I} \to \mathbb{R}^n$, with corresponding change of variables $\phi : I \to \tilde{I}$ (in particular, $\gamma(t) = \tilde{\gamma}(\phi(t))$ for all $t \in I$), then

$$\kappa_\gamma(t) = \kappa_{\tilde{\gamma}}(\phi(t)), \qquad t \in I.$$

(a) To compute the curvature of ℓ, we first calculate

$$\ell'(t) = \mathbf{v}, \qquad |\ell'(t)| = |\mathbf{v}|, \qquad \frac{\ell'(t)}{|\ell'(t)|} = \frac{\mathbf{v}}{|\mathbf{v}|}.$$

Since all three of the above are constant, we see from our given definition that

$$\kappa_\ell(t) = \frac{1}{|\ell'(t)|} \left| \left(\frac{\ell'}{|\ell'|} \right)' (t) \right| = \frac{1}{|\mathbf{v}|} \left| \frac{d}{dt} \left(\frac{\mathbf{v}}{|\mathbf{v}|} \right) \right| = 0, \qquad t \in \mathbb{R}.$$

(b) First, some direct computations yield

$$\gamma_R'(t) = (-R\sin t, R\cos t), \qquad |\gamma_R'(t)| = R, \qquad \frac{\gamma_R'(t)}{|\gamma_R'(t)|} = (-\sin t, \cos t).$$

Applying our given definition to the above then yields, for any $t \in \mathbb{R}$,

$$\kappa_{\gamma_R}(t) = \frac{1}{R}\left|\frac{d}{dt}(-\sin t, \cos t)\right| = \frac{1}{R}|(-\cos t, -\sin t)| = \frac{1}{R}.$$

(c) Recalling the relation between γ' and $\tilde{\gamma}'$ (see lectures or the lecture notes), we have that

$$\begin{aligned}
\kappa_\gamma(t) &= \frac{1}{|\gamma'(t)|}\left|\left(\frac{\gamma'}{|\gamma'|}\right)'(t)\right| \\
&= \frac{1}{|\phi'(t)||\tilde{\gamma}'(\phi(t))|}\left|\left[\frac{\phi' \cdot \tilde{\gamma}'(\phi)}{|\phi'| \cdot |\tilde{\gamma}'(\phi)|}\right]'(t)\right| \\
&= \frac{1}{|\phi'(t)||\tilde{\gamma}'(\phi(t))|}\left|\left[\frac{\tilde{\gamma}'(\phi)}{|\tilde{\gamma}'(\phi)|}\right]'(t)\right|,
\end{aligned}$$

where $\tilde{\gamma}'(\phi)$ denotes the composition of $\tilde{\gamma}'$ with ϕ, and where the last step follows from the observation that $|\phi'(t)|^{-1}\phi'(t)$ has constant value ± 1.

Furthermore, applying the chain rule to the outer derivative in the above, we see that

$$\begin{aligned}
\kappa_\gamma(t) &= \frac{1}{|\phi'(t)||\tilde{\gamma}'(\phi(t))|}\left|\left(\frac{\tilde{\gamma}'}{|\tilde{\gamma}'|}\right)'(\phi(t)) \cdot \phi'(t)\right| \\
&= \frac{1}{|\tilde{\gamma}'(\phi(t))|}\left|\left(\frac{\tilde{\gamma}'}{|\tilde{\gamma}'|}\right)'(\phi(t))\right| \\
&= \kappa_{\tilde{\gamma}}(\phi(t)),
\end{aligned}$$

as desired. (In the first step, we applied the chain rule to the function $|\tilde{\gamma}'|^{-1} \cdot \tilde{\gamma}'$.)

(8) *(Polar curves)* Let $h : \mathbb{R} \to \mathbb{R}$ be a smooth positive periodic function, with period 2π:

$$h(\theta) > 0, \qquad h(\theta + 2\pi) = h(\theta), \qquad x \in \mathbb{R}.$$

Let the *polar curve* P be the set of all points in $\mathbb{R}^2$ satisfying the relation

$$r = h(\theta)$$

in polar coordinates. (Here, you can assume that P is indeed a curve.)

- **(a)** The unit circle $\mathcal{C} = \{(x, y) \in \mathbb{R} \mid x^2 + y^2 = 1\}$ is a polar curve. What is h here?
- **(b)** Give a injective parametrisation of P whose image is all of P except for a single point.

(c) Derive a formula for the arc length of P.

(a) Since the expression $x^2 + y^2$ is simply r^2 in polar coordinates, it follows that $\mathcal{C}$ is the polar curve $r = 1$, that is, with $h(\theta) = 1$ for all $\theta \in \mathbb{R}$.

(b) The most straightforward method is to set the parameter t to be the polar angle θ. (This is analogous to the polar parametrisation of the unit circle). Since

$$x = r \cos\theta = h(\theta)\cos\theta, \qquad y = r \sin\theta = h(\theta)\sin\theta,$$

and since the above starts repeating itself after θ has increased by 2π, then the following is an injective parametrisation of P whose image misses only a single point of P:

$$\gamma : (0, 2\pi) \to P, \qquad \gamma(t) = (h(t)\cos t,\ h(t)\sin t).$$

(In particular, γ misses the point $\gamma(0) = (h(0),\ 0)$.

(c) We use the parametrisation in part (b). Note that

$$|\gamma'(t)| = |(h'(t)\cos t - h(t)\sin t,\ h'(t)\sin t + h(t)\cos t)| = \sqrt{[h'(t)]^2 + [h(t)]^2}.$$

As a result, the arc length of P is

$$L(P) = L(\gamma) = \int_0^{2\pi} \sqrt{[h(t)]^2 + [h'(t)]^2}\, dt.$$

(See also Question (5c) of Problem Sheet 5.)

(9) *(Conic sections)* Let N denote the following cone:

$$N = \{(x, y, z) \in \mathbb{R}^3 \mid z^2 = x^2 + y^2\}.$$

In addition, let $P \subseteq \mathbb{R}^3$ denote an arbitrary plane that does not pass through the origin. More specifically, P is a set of the form

$$P = \{(x, y, z) \in \mathbb{R}^3 \mid ax + by + cz = d\}$$

where $a, b, c, d \in \mathbb{R}$ satisfy $(a, b, c) \neq (0, 0, 0)$ and $d \neq 0$. A set of the form $N \cap P$ (i.e. the intersection of the cone N and the plane P) is called a *conic section*.

(a) Use the theorem in Question (9) of Problem Sheet 4 to show that any conic section $N \cap P$ is indeed a curve. *(Hint: You will have to be resourceful to do this. The first step is to express* $N \cap P$ *as an appropriate level set.)*

(b) Find examples of such planes P such that the conic section $N \cap P$ is:

(i) A circle.

(ii) An ellipse.

(iii) A parabola.

(iv) A hyperbola.

Check your answers graphically on a computer!

(a) We begin by observing that $N \cap P$ can be expressed as a level set

$$N \cap P = \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = 0,\ g(x, y, z) = d\}, \tag{1}$$

where the functions f and g are given by

$$f : \mathbb{R}^3 \to \mathbb{R}, \qquad f(x, y, z) = x^2 + y^2 - z^2,$$
$$g : \mathbb{R}^3 \to \mathbb{R}, \qquad g(x, y, z) = ax + by + cz.$$

Taking the gradients of f and g yields, for any $(x, y, z) \in \mathbb{R}^3$,

$$\nabla f(x, y, z) = (2x,\ 2y,\ -2z)_{(x,y,z)}, \qquad \nabla g(x, y, z) = (a,\ b,\ c)_{(x,y,z)}.$$

Recall that $\nabla f(x, y, z) \times \nabla g(x, y, z)$ vanishes if and only if the two gradients point in the same or in the opposite directions. Since $(a, b, c) \neq (0, 0, 0)$ by assumption, it follows that $\nabla f(x, y, z) \times \nabla g(x, y, z)$ vanishes if and only if there is some $\lambda \in \mathbb{R}$ such that

$$(x, y, -z) = \lambda(a, b, c).$$

We now let L denote the set

$$L = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y, -z) = \lambda(a, b, c) \text{ for some } \lambda \in \mathbb{R}\}.$$

(Note that L is a line through the origin.) We claim that *no point of* L *lies in* $N \cap P$. To show this, we suppose $(x, y, z) \in L$—so that $(x, y, -z) = \lambda(a, b, c)$ for some $\lambda \in \mathbb{R}$—and we assume (x, y, z) does lie in $N \cap P$. Then, (1) implies that

$$x^2 + y^2 - z^2 = 0, \qquad x^2 + y^2 - z^2 = \lambda(ax + by + cz) = \lambda d. \tag{2}$$

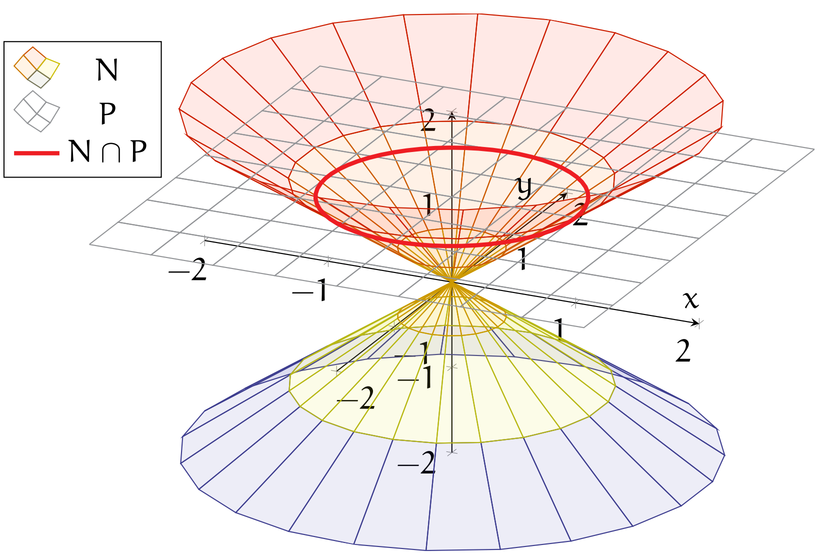
Since $d \neq 0$ by assumption, (2) yields $\lambda = 0$. However, this implies $(x, y, z) = (0, 0, 0)$, which by assumption does not lie in $N \cap P$, resulting in a contradiction. This proves the claim.

Combining all the above, we deduce that $\nabla f(x, y, z) \times \nabla g(x, y, z)$ does not vanish at any point of $N \cap P$. Therefore, applying the theorem in Question (9) of Problem Sheet 4 to $N \cap P$ in (1), we conclude that it is indeed a curve.

(b) (i) One example of a circle is to take $N \cap P$, with

$$P = \{(x, y, z) \in \mathbb{R}^3 \mid z = 1\}.$$

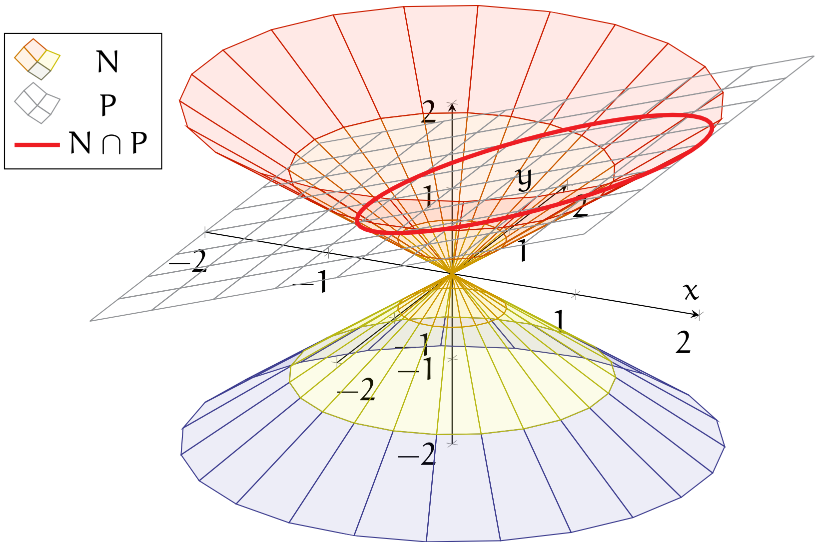
(In general, any P with $(a, b) = (0, 0)$ suffices.) This is illustrated below:



(ii) One example of a (non-circular) ellipse is to take $N \cap P$, with

$$P = \left\{(x, y, z) \in \mathbb{R}^3 \mid z = \frac{1}{2}x + 1\right\}.$$

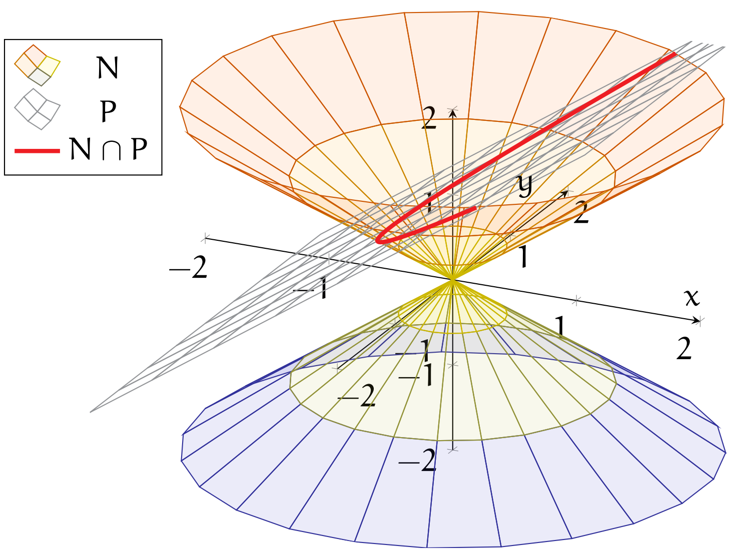
(In general, any P with $a^2 + b^2 < c^2$ suffices.) This is illustrated below:



(iii) One example of a parabola is to take $N \cap P$, with

$$P = \left\{(x, y, z) \in \mathbb{R}^3 \mid z = x + 1\right\}.$$

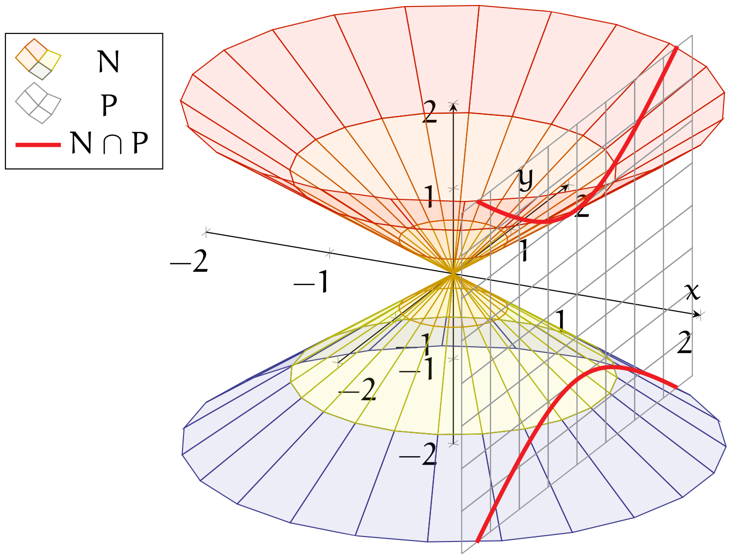
(In general, any P with $a^2 + b^2 = c^2$ suffices.) This is illustrated below:



(iv) One example of a hyperbola is to take $N \cap P$, with

$$P = \left\{(x, y, z) \in \mathbb{R}^3 \mid x = 1\right\}.$$

(In general, any P with $a^2 + b^2 > c^2$ suffices.) This is illustrated below:


N
P
N ∩ P
x
y
2
1
−1
−2