Today

- Final material on game theory
  (examinable)

Revision

Week 4 - extreme point solutions
basic feasible solutions

Week 10/11 Game theory

General 2-player games

Example 12.1. Suppose that Rosemary and Colin are working on a joint project. Each of them can choose to "work hard" or "goof off." Both of them must work hard together to receive a high mark for the project. Both have utility 3 for receiving a high mark utility 1 for goofing off (regardless of what mark they receive) and utility 0 for working hard but not receiving a high mark. Give the payoff matrix for this game.

Colin

Rosemary	work hard (w)	goof off (g)
work hard (w)	(3, 3)	(0, 1)
goof off (g)	(1, 0)	(1, 1)

not zero-sum game

Rosemary's set of strategies $R = \{w, g\}$

Colin's set of strategies $C = \{w, g\}$

Write out Rosemary's payoff function $u_1 : R \times C \to \mathbb{R}$

Write out Colin's payoff function $u_2 : R \times C \to \mathbb{R}$

$u_1(w,w) = 3$ $\qquad$ $u_2(w,w) = 3$

$u_1(w,g) = 0$ $\qquad$ $u_2(w,g) = 1$

$u_1(g,w) = 1$ $\qquad$ $u_2(g,w) = 0$

$u_1(g,g) = 1$ $\qquad$ $u_2(g,g) = 1$

For general 2-player game

Let R = set of Rosemary's strategies

C = set of Colin's strategies

For $r \in R$ and $c \in C$, (r, c) is a Nash equilibrium if neither player has an incentive to change strategies (assuming the other player doesn't change strategy) i.e.

write this in symbols assuming u_1 is Rosemary's payoff function
u_2 is Colin's payoff function

$$u_1(r, c) \geq u_1(r', c) \quad \forall r' \in R$$
$$u_2(r, c) \geq u_2(r, c') \quad \forall c' \in C$$

Does our example have any Nash equilibria?

Yes

Colin

Rosemary	work hard (w)	goof off (g)
work hard (w)	(3, 3)	(0, 1)
goof off (g)	(1, 0)	(1, 1)

(w, w) is pure Nash equilibrium

(g, g) is pure Nash equilibrium

(w, g) and (g, w) are not Nash equilibria.

How can we systematically and quickly find all pure Nash equilibrium.

Method 1: check each $(r,c) \in R \times C$. Quite slow

Example 12.2. Find all pure Nash equilibria for the games with the following payoff matrices.

Colin

Rosemary

	c_1	c_2	c_3
r_1	$(0,-1)$	$(1,2)$	$(2,-1)$
r_2	$(2,1)$	$(0,-1)$	$(2,1)$
r_3	$(0,2)$	$(-1,1)$	$(1,-1)$

	c_1	c_2	c_3
r_1	$(1,0)$	$(0,1)$	$(1,-1)$
r_2	$(-1,1)$	$(1,0)$	$(0,1)$

No pure Nash equilibria.

3 pure Nash equilibria: (r_1,c_2), (r_2,c_1), (r_2,c_3)

(r_i,c_j) is a (pure) Nash equilibrium if and only if

c_j gives highest payoff to Colin when Rosemary plays r_i (we say c_j is Colin's best response to r_i)

<u>and</u>

r_i gives highest payoff to Rosemary when Colin plays c_j (we say r_i is Rosemary's best response to c_j).

Method: mark each player's best response to the other player's strategies.
(r_i,c_j) is a Nash equilibrium if and only if it is marked twice

Not all general 2-player games have a pure Nash equilibria

$R = \{r_1, \dots, r_k\}$ Roseman's strategies
$C = \{c_1, \dots, c_l\}$ Colin's strategies

$$\Delta(R) = \{ \underline{x} = (x_1, \dots, x_k) : x_1 + x_2 + \dots + x_k = 1 \}$$
$$\Delta(C) = \{ \underline{y} = (y_1, \dots, y_l) : y_1 + y_2 + \dots + y_l = 1 \}$$

Recall defn: consider a zero-sum game with payoff matrix A.
For $\underline{x} \in \Delta(R)$ and $\underline{y} \in \Delta(C)$, $(\underline{x}, \underline{y})$ is a mixed Nash equilibrium if

$$\underline{x}^T A \underline{y} \geq \underline{x}'^T A \underline{y} \quad \forall \underline{x}' \in \Delta(R)$$
and
$$\underline{x}^T A \underline{y} \leq \underline{x}^T A \underline{y}' \quad \forall \underline{y}' \in \Delta(C)$$

Defn Consider a general game with payoff matrix A_1 for Roseman and A_2 for Colin.
For $\underline{x} \in \Delta(R)$ and $\underline{y} \in \Delta(C)$, $(\underline{x}, \underline{y})$ is a mixed Nash equilibrium if

$$\underline{x}^T A_1 \underline{y} \geq \underline{x}'^T A_1 \underline{y} \quad \forall \underline{x}' \in \Delta(R)$$
$$\underline{x}^T A_2 \underline{y} \geq \underline{x}^T A_2 \underline{y}' \quad \forall \underline{y}' \in \Delta(C)$$

John Nash proved that every general 2-player game has a mixed Nash equilibrium using Brouwer's fixed point theorem from topology. No easy way of finding these mixed Nash equilibrium.

See exam info document on QMplus
(update by Monday)

Recap quiz (paraphrased defns/theorems)

Consider an LP in standard equation form

$$\begin{aligned} &\text{maximise} \quad \underline{c}^T \underline{x} \\ &\text{subject to} \quad A\underline{x} = \underline{b}, \quad \underline{x} \geq 0. \end{aligned}$$

An extreme point solution is a feasible solution $\underline{x}$ such that $\underline{x}$ cannot be written as $\lambda \underline{y} + (1-\lambda)\underline{z}$ where $\underline{y}, \underline{z}$ are distinct feasible solutions and $\lambda \in (0,1)$.

A basic feasible solution is a feasible solution $\underline{x}$ in which the non-zero entries of $\underline{x}$ correspond to linearly independent columns of A.

Last time we proved two results

① Every LP (in standard equation form) has an optimal solution that is an extreme point solution (provided it has at least one optimal solution).

② Given an LP in standard equation form every basic feasible solution is an extreme point solution and vice versa.

(proof not completed)

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(b) Consider the following linear program in standard equation form:

$$\begin{array}{llr} \text{maximise} & x_1 + 2x_2 - 3x_3 + 7x_5 & \\ \text{subject to} & x_1 + 2x_2 + 2x_3 + x_4 & = 3, \\ & x_1 + 2x_2 + 7x_3 \quad + x_5 & = 3, \\ & 2x_1 + 4x_2 + 7x_3 \quad\quad + x_6 & = 6, \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 & \end{array}$$

For each of the following values of $\mathbf{x}^\mathsf{T} = (x_1, x_2, x_3, x_4, x_5, x_6)$ say whether or not this value is a **basic feasible solution** of this linear program and also whether or not it is an **extreme point solution** of this linear program. Justify your answers. [**9**]

(i) $\mathbf{x}^\mathsf{T} = (1, 1, 0, 0, 0, 0)$

(ii) $\mathbf{x}^\mathsf{T} = (1, 0, 0, 2, 2, 4)$

(iii) $\mathbf{x}^\mathsf{T} = (0, 0, 0, 3, 3, 6)$

Here $A = \begin{pmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 2 & 7 & 0 & 1 & 0 \\ 2 & 4 & 7 & 0 & 0 & 1 \end{pmatrix}$

(iii) This is a BFS.

First its clear that all the constraints and sign restrictions are satisfied

Also we must check that $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$ are linearly independent. because they are the standard basis for $\mathbb{R}^3$.

[Alternatively if we solve $\lambda_1\begin{pmatrix}1\\0\\0\end{pmatrix} + \lambda_2\begin{pmatrix}0\\1\\0\end{pmatrix} + \lambda_3\begin{pmatrix}0\\0\\1\end{pmatrix} = \underline{0}$ gives $\lambda_1 = \lambda_2 = \lambda_3 = 0$, which shows three vectors are linearly independent.]

Hence it is also an extreme point solution by a theorem in lectures.

(i) $(1,1,0,0,0,0)'$ satisfies constraints and sign restrictions so is feasible.

By definition, $(1,1,0,0,0,0)$ is a BFS if and only if $\begin{pmatrix}1\\1\\2\end{pmatrix}$ and $\begin{pmatrix}2\\2\\4\end{pmatrix}$ are linearly independent.

However these two vectors are linearly dependent because $2\begin{pmatrix}1\\1\\2\end{pmatrix}-\begin{pmatrix}2\\2\\4\end{pmatrix}=\underline{0}$ so this is not a BFS.

This is not an extreme point solution (by a theorem from lectures $\underline{x}$ is an extreme point solution if and only if it is a BFS).

(ii) This is not a BFS because the vectors $\begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$ are linearly dependent since $\begin{pmatrix}1\\1\\2\end{pmatrix}-\begin{pmatrix}1\\0\\0\end{pmatrix}-\begin{pmatrix}0\\1\\0\end{pmatrix}-2\begin{pmatrix}0\\0\\1\end{pmatrix}=\underline{0}$.

Also not an extreme point solution by the same theorem from lectures (in part (i)).

[Alternatively: have more vectors than the dimension of the vectors, so linearly dependent].

(c) Consider an arbitrary linear program in standard equation form:

$$\begin{aligned} \text{maximise} \quad & \mathbf{c}^{\mathsf{T}}\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Suppose that $\mathbf{x}$ is an optimal solution to this linear program. Show that if $\mathbf{x}$ is **not** an extreme point solution then we can express $\mathbf{x}$ as $\mathbf{x} = \lambda\mathbf{y} + (1-\lambda)\mathbf{z}$ where $\lambda \in (0, 1)$ and $\mathbf{y}$ and $\mathbf{z}$ are two different optimal solutions of this program. [8]

This is a bookwork question

This is claim 1 from long proof in week 4.

Basic idea.

$\underline{x}$ is optimal so $\underline{x}$ is feasible.

$\underline{x}$ is not extreme point solution means by definition that $\underline{x}$ can be written as $\underline{x} = \lambda \underline{y} + (1-\lambda)\underline{z}$ for distinct feasible solutions $\underline{y}$ and $\underline{z}$ and $\lambda \in (0,1)$.

Then show $\underline{y}$ and $\underline{z}$ are optimal

Get credit for clearly saying what you're trying to do even if you can't actually do it.

Basic terminology

strategy, outcome, payoff matrix, zero-sum game, mixed strategy

choice, pair of strategies, payoff for each outcome, vector of probabilities.

For zero sum games

What is a pure Nash equilibrium in words/symbols?

What is the security level of a strategy in words/symbols?

What is best security level for a player?

How are they related?

Thm

(r_p, c_q) is a pure Nash equilibrium iff

security of r_p = security of c_q.

For zero sum games
How do we compute expected payoff
when players use mixed strategy

If Rose may plays $\underline{x} \in \Delta(R)$ and Colin plays $\underline{y} \in \Delta(C)$
expected payoff $\underline{x}^T A \underline{y} = \sum_{i,j} a_{ij} x_i y_j$

What is a mixed Nash equilibrium in words/symbols?
What is the security level of a mixed strategy
in words/symbols

$(\underline{x}, \underline{y})$ is mixed Nash equilibrium if neither player has an incentive to change

Security of $\underline{x}$ is smallest payoff to Rose may if she plays $\underline{x}$ and Colin plays a pure strategy

$$= \min_j \underline{x}^T A e_j = \min_j \underline{\underline{x}^T A}$$

How are they related.
How do we write LP's to find optimal mixed strategy?
i.e. mixed strategy with best security.

Question 5 [24 marks].

(a) In the following question, let $\beta \in \mathbb{R}$ be a fixed constant. Suppose a zero-sum 2-player game has the following payoff matrix, given from the perspective of the row player:

	1	2
1	β	6
2	-6	0

(i) Suppose that $\beta = 0$. Give the security levels for each of the row and column players' strategies. List all pure Nash equilibria for this game or explain why the game does not have a pure Nash equilibrium. **[4]**

(ii) For what range of possible values for β is $(1,2)$ a pure Nash equilibrium for this game? Justify your answer. **[6]**

(iii) For what range of possible values for β does this game have a general Nash equilibrium? Justify your answer. **[4]**

(i)

	1	2
1	0	6
2	-6	0

Security level for row player's strategy 1 $= 0$

2 $= -6$

Column player's strategy 1 $= 0$

2 $= 6$

The outcome $(1,1)$ is a pure Nash equilibrium because the security levels match.
No other pure Nash equilibria.

(ii)

	1	2
1	β	6
2	-6	0
		6

$(1,2)$ is a pue a pure Nash equilibrium if and only if

security level when Colin plays 2 $=$ security level when Rosemary plays 1

This is 6 no what β is

This $\min(\beta, 6)$

We have $\min(\beta,6)=6$ if and only if $\underline{\beta \geq 6}$.

(iii) All values of β because by thm from lectures all 2-player zero-sum games have a mixed Nash equilibrium.

(b) Consider the following 2-player game. Rosemary and Colin each select a number n from the set $\{1, 2, 3\}$. If they choose the same number, neither player wins anything. Otherwise, if the sum of their numbers is at least 5, both of them win £1. Finally, if their numbers do not match and do not sum to at least 5, then the player who selected the largest number n wins £n and the other player loses £n.

(i) Give the payoff matrix for this game (as usual, suppose that Rosemary is the row player and give her payoff first in each cell). **[4]**

(ii) Is this a zero sum game? Justify your answer. **[2]**

(iii) List all pure Nash equilibria for this game. **[4]**

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(b) Consider the 2-player zero-sum game with the following payoff matrix (which is given, as usual, from the perspective of the row player).

		5/6	1/6
		c_1	c_2.
1/3	r_1	6	-6
2/3	r_2	3	9

(i) Write a linear program that finds the optimal mixed strategy for the row player (i.e. the mixed strategy with the best security level). You do not have to solve this linear program. [**6**]

(ii) Consider the mixed strategy $\mathbf{x}$ for the row player and $\mathbf{y}$ for the column player given by $\mathbf{x}^{\mathsf{T}} = (1/3, 2/3)$ and $\mathbf{y}^{\mathsf{T}} = (5/6, 1/6)$. Show that this pair of strategies is a mixed Nash equilibrium for this game. [**8**]

(ii) It is enough to show that the security level of $\underline{x}$ is equal to security level of $\underline{y}$.

expected payoff when Rosemary plays $\underline{x}$, Colin plays c_1

$\frac{1}{3} \times 6 + \frac{2}{3} \times 3 = 4$

Rosemary plays $\underline{x}$, Colin plays c_2

$\frac{1}{3} \times -6 + \frac{2}{3} \times 9 = 4$

Security level of $\underline{x}$ = $\min(4, 4) = 4$

expected payoff when Colin plays $\underline{y}$, Rosemary plays r_1

$5/6 \times 6 + \frac{1}{6} \times (-6) = 4$

Colin plays $\underline{y}$, Rosemary plays r_2

$5/6 \times 3 + 1/6 \times 9 = 4$

Security for $\underline{y}$ = $\max(4, 4) = 4$

Security levels match and so $(\underline{x}, \underline{y})$ is by a theorem in lectures.