Actuarial Mathematics II
MTH5125

Revision 2

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Spring Term

Plan

- Multiple States/Transitions
- Multiple Decrements
- Projected Future Cashflow Techniques

Multiple States

- The occupancy probability -probability that (x) now in state i will remain in state i for t years can be computed using:

$$ {}_tp_x^{\overline{ii}} = \exp\left(-\int_0^t \sum_{j=0, j\neq i}^{m} \mu_{x+s}^{ij} ds \right) $$

Multiple States

Kolmogorov forward equations:

$$ {}_{t+h}p_x^{ij} = {}_tp_x^{ij}\ {}_hp_{x+t}^{jj} + \sum_{k=0,k\neq j}^{m} {}_tp_x^{ik}\ {}_hp_{x+t}^{kj} $$

$$ {}_hp_{x+t}^{jj} = 1 - h\sum_{k=0,k\neq j}^{m} \mu_{x+t}^{jk} + o(h) $$

$$ {}_hp_{x+t}^{kj} = h\mu_{x+t}^{kj} + o(h) $$

Multiple States

EPV of continuous state dependent time annuity:

$$\bar{a}_x^{ij} = \int_0^\infty e^{-\delta t} \ {}_t p_x^{ij} \, dt$$

If the annuity is payable at the start of each year, from the current time, conditional on the life being in state j, given that the life is currently in state i.
The EPV of a discrete anuity due is:

$$\ddot{a}_x^{ij} = \sum_{k=0}^\infty v^k \ {}_k p_x^{ij}$$

If $i \neq j \ {}_0 p_x^{ij} = 0$

State dependent annuities

The EPV of a term anuity due is:

$$\ddot{a}^{ij}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k \ {}_k p^{ij}_x$$

$$\bar{a}^{ij}_{x:\overline{n}|} = \bar{a}^{ij}_x - e^{-\delta n} \sum_{k=0}^{m} \ {}_n p^{ik}_x \ \bar{a}^{kj}_{x+n}$$

For example: alive-dead model: $\bar{a}^{00}_{x:\overline{n}|} = \bar{a}^{00}_x - e^{-\delta n} \ {}_n p^{00}_x \ \bar{a}^{00}_{x+n}$
Sickness-death model: $\bar{a}^{00}_{x:\overline{n}|} = \bar{a}^{00}_x - e^{-\delta n} \ {}_n p^{00}_x \ \bar{a}^{00}_{x+n} - e^{-\delta n} \ {}_n p^{01}_x \ \bar{a}^{10}_{x+n}$

State dependent insurance benefits

Suppose a unit benefit is payable immediately on each future transfer into state j, given that the life is currently in state i (which may be equal to j). Then the expected present value of the benefit is:

$$\bar{A}_x^{ij} = \int_0^{\infty} \sum_{k=0, k\neq j}^{\infty} e^{-\delta t}\ {}_t p_x^{ik}\ \mu_{x+t}^{kj} dt$$

State dependent insurance benefits

Note that this benefit does not require the transition to be directly from state i to state j, and if ther is a posibility of transitions into state j than it values a benefit of 1 paid each time the life transitions into state j.

$$\begin{aligned} \bar{A}^{ij}_{x:\overline{n}|} &= \int_0^n \sum_{k=0,k\neq j}^{m} e^{-\delta t} \;{}_t p_x^{ik} \; \mu_{x+t}^{kj} dt \\ &= \bar{A}_x^{ij} - e^{-\delta n} \sum_{k=0}^{m} {}_n p_x^{ik} \bar{A}_{x+n}^{kj} \end{aligned}$$

State dependent insurance benefits

Alive- dead model:

$$\bar{A}^{00}_{x:\overline{n}|} = \bar{A}^{00}_{x} - e^{-\delta n} \; {}_{n}p^{00}_{x} \bar{A}^{00}_{x+n}$$

Sickness-death model for $i = 0$ and $j = 1$:

$$\bar{A}^{01}_{x:\overline{n}|} = \bar{A}^{01}_{x} - e^{-\delta n} \; {}_{n}p^{00}_{x} \bar{A}^{01}_{x+n} - e^{-\delta n} \; {}_{n}p^{01}_{x} \bar{A}^{11}_{x+n}$$

Premiums

- We assume that premiums are calculated using the equivalence principle and that lives are in state 0 at the policy inception date.
- Premiums are calculated by solving the equation of value using the appropriate anuity and insurance functions

Policy values

The state j policy value time t,denoted ${}_tV^{(j)}$ is the expected value at time t of the future loss random variable for a policy which is in State j at time t.

$$
\begin{aligned}
{}_tV^{(j)} &= EPV \text{ at } t \text{ of future benefits } + \text{expenses} \\
&\quad -EPV \text{ at } t \text{ of future premiums,} \\
&\quad \text{given the insured is in state } j \text{ at } t
\end{aligned}
$$

Policy values

The general form of Thiele's equations:

$$\frac{d}{dt}V^i(t) = V^i(t)\delta(t) - b_i(t)\sum_{j\neq i}\mu_{x+t}^{ij}\left(b_{ij}(t) + V^j(t) - V^i(t)\right)$$

Note that this is a system of simultaneous ODEs, one for each state i. If, as is usually the case, benefits and force of interest do not depend on t, we get a simpler system:

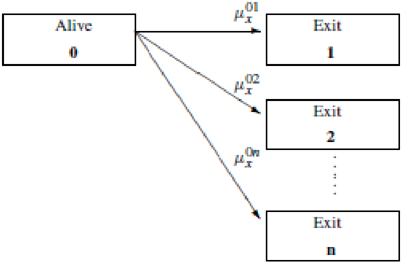
$$\frac{d}{dt}V^i(t) = V^i(t)\delta - b_i\sum_{j\neq i}\mu_{x+t}^{ij}\left(b_{ij} + V^j(t) - V^i(t)\right)$$

Multiple Decrement Model

Multiple decrement models are extensions of standard mortality models with simultaneous several causes of decrement.

A multiple decrement model is a multi-state model with:

- one active state (initial, State 0)
- one or more **absorbing** exit states.

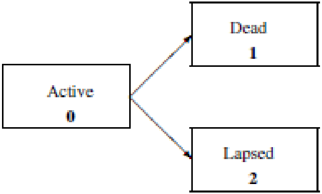


Multiple Decrement Probabilities

I will introduce equivalent notations from the DHW textbook and the accepted (in practice) UK notations

- Dependent survival probability (): ${}_tp_x^{00}$ or ${}_t(ap)_x$
- Dependent transition probability: ${}_tp_x^{0j}$ or ${}_t(aq)_x^j$
- Dependent total transition probability: ${}_tp_x^{0\bullet}$ or ${}_t(aq)_x$
- Forces of transition: μ_{x+t}^{0j} or μ_{x+t}^j
- Total force of transition $\mu_{x+t}^{0\bullet}$ or $(a\mu)_{x+t}$
- Multiple decrement table
  - Active lives: l_x or $(al)_x$
  - Decrements: $d_x^{(j)}$ or $(ad)_x^{(j)}$

Multiple Decrement Table: one year relations in an example



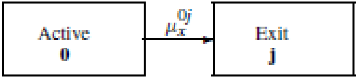
$$p_x^{00} + p_x^{01} + p_x^{02} = 1$$

or in the UK notation:

$$(ap)_x + (aq)_x^1 + (aq)_x^2 = 1$$

Independent Single Decrement Example

A simpler assumption: One exit state j with the same transition intensity into the state: independent single decrement model for decrement j



Reduced two state model with:
Independent survival probability denoted as:

$$ {}_t p_x^{*(j)} \equiv {}_t p_x^{j} = \exp\left(-\int_0^t \mu_{x+u}^{0j} du \right) $$

and

Independent Single Decrement Example

Independent transition probability:

$$ {}_t q_x^{*(j)} \equiv {}_t q_x^j = \int_0^t {}_s p_x^{*(j)} \mu_{x+s}^{0j} ds $$

Note that the second notation refers to the UK notation.

Similar to the life table functions l_x and d_x for the alive- dead model we will derive a multiple decrement table
Let l_{x_0} be the radix of the table at the initial age x_0
Define

$$l_{x+t} = l_{x_0}\ {}_t p_{x_0}^{00}$$

and for $j = 1, 2, ...m$ and $x \geq x_0$,

$$d_x^{(j)} = l_x p_x^{0j}$$

l_x - the expected number of active lives (in state 0) at age x out of l_{x_0}

$d_x^{(j)}$- the expected number of lives exiting by mode of decrement j in the year of age x to $x+1$

Fractional age with UDD

UDD: here it means uniform distribution of decrements
For $0 \leq t \leq 1$ and integer x and for each exit mode j assume that for $j \neq 0$:

$$ {}_t p_x^{0j} = t \times p_x^{0j} $$

The exits from the starting state are uniformly spread over each year.

Fractional age with constant transition forces

For $0 \leq t \leq 1$ and integer x assume that for each exit mode j, μ^{0j}_{x+t} is a constant for each age x and equal to $\mu^{0j}(x)$:

$$\mu^{0\bullet}(x) = \sum_{k=1}^{m} \mu^{0k}(x)$$

$\mu^{0\bullet}(x)$ is the total force of transition out of state 0 at age $x + t$ for $0 \leq t < 1$ and:

$${}_tp^{00}_x = \exp\left(-\mu^{0\bullet}(x)\, t\right)$$

Note that the total force of transition out of state 0 for the year of age x to $x + 1$.

$$\begin{aligned} p^{0\bullet}_x &= 1 - p^{00}_x = \sum_{k=1}^{m} p^{0k}_x \\ &= 1 - \exp\left(-\mu^{0\bullet}(x)\right) \end{aligned}$$

Constant transition forces

Then for integer x and for $0 \leq t < 1$, we have:

$$ {}_t p_x^{0j} = \int_0^t {}_r p_x^{00} \mu_{x+r}^{0j} dr = \int_0^t e^{-r\mu^{0\bullet}(x)} \mu^{0j}(x)\, dr $$

hence:

$$ {}_t p_x^{0j} = \frac{\mu^{0j}(x)}{\mu^{0\bullet}(x)} \left(1 - e^{-t\mu^{0\bullet}(x)}\right) = \frac{\mu^{0j}(x)}{\mu^{0\bullet}(x)} \left(1 - \left(p_x^{00}\right)^t\right) $$

Let $t \rightarrow 1 : \frac{\mu^{0j}(x)}{\mu^{0\bullet}(x)} = \frac{p_x^{0j}}{p_x^{0\bullet}}$ and hence:

$$ {}_t p_x^{0j} = \frac{p_x^{0j}}{p_x^{0\bullet}} \left(1 - \left(p_x^{00}\right)^t\right) $$

Constant transition forces

We can also use:

$$ {}_t p_x^{0j} = \frac{\mu^{0j}(x)}{\mu^{0\bullet}(x)} \left(1 - e^{-t\mu^{0\bullet}(x)}\right) $$

For the year of age x to $x+1$:

$$ p_x^{0j} = \frac{\mu_x^{0j}}{\mu_x^{0\bullet}} \left(1 - e^{-\mu_x^{0\bullet}}\right) = \frac{\mu_x^{0j}}{\mu_x^{0\bullet}} p_x^{0\bullet} $$

Note that the decrement j independent survival probability is:

$$p_x^{*(j)} = \exp\left(-\int_0^1 \mu_{x+t}^{0j} dt\right) = \exp\left(\log\left(p_x^{00}\right)^{\frac{p_x^{0j}}{p_x^{0\bullet}}}\right)$$

Hence:

$$p_x^{*(j)} = \left(p_x^{00}\right)^{\frac{p_x^{0j}}{p_x^{0\bullet}}} \quad (9.8)$$

$$p_x^{0j} = \frac{\log p_x^{*(j)}}{\log p_x^{00}} p_x^{0\bullet}$$

With$p_x^{00} = \prod_{j=1}^{m} p_x^{*(j)}$ and $p_x^{0\bullet} = 1 - p_x^{00}$

How did we get that?

Assume UDD each decrement is uniformly distributed in the multiple decrement model and for an integer x and $0 \leq t < 1$:

$$
\begin{aligned}
{}_t p_x^{0k} &= t p_x^{0k} \\
{}_t p^{00} &= 1 - t p_x^{0\bullet}
\end{aligned}
$$

$$
{}_t p_x^{00} \mu_{x+t}^{0j} = p_x^{0j}
$$

or:

$$
\mu_{x+t}^{0j} = \frac{p_x^{0j}}{1 - t p_x^{0\bullet}}
$$

How did we get that?

Integrating on both sides:

$$\begin{aligned}
\int_0^1 \mu_{x+t}^{0j} dt &= \frac{p_x^{0j}}{p_x^{0\bullet}} \left(-\log\left(1 - p_x^{0\bullet}\right)\right) \\
&= \frac{p_x^{0j}}{p_x^{0\bullet}} \left(-\log\left(p_x^{00}\right)\right) \\
&= -\log\left(p_x^{00}\right)^{\frac{p_x^{0j}}{p_x^{0\bullet}}}
\end{aligned}$$

which we substitute in $p_x^{*(j)} = \exp\left(-\int_0^1 \mu_{x+t}^{0j} dt\right)$.

Profit testing

The purpose of a profit test is to identify the profit which the insurer can claim from the contract at the end of each time period

Profit test basis: assumptions about:

- the expenses which will be incurred,
- the survival model for the policyholder,
- the rate of interest to be earned on cash flows within each time period before the profit is released
- other items such as an assessment of the probability that the policyholder surrenders the policy.

Profit testing

Typical feature of net cash flows: several of the net cash flows in later years are negative.

The reserve may be equal to the policy value, or may be some different amount.

Profit vector

Profit at t :

$$Pr_t = (1+i)\left(P + {}_{t-1}V - E_t\right) - Sq_{x+t-1} - {}_tVp_{x+t-1}$$

Alternative way of writing the profit:

$$\text{Pr}_t = (1+i)\left(P - E_t\right) + \Delta V_t - Sq_{x+t-1}$$

where $\Delta V_t = (1+i)\ {}_{t-1}V - {}_tVp_{x+t-1}$ is the change in reserve in year t.

The vector $Pr = (Pr_0, ..., Pr_t)'$ is called the profit vector for the contract.

Reserves: Profit signature

Multiplying Pr_t by $_{(t-1)}p_x$ gives a vector each of whose elements is the expected profit at the end of each year given only that the contract was in force at age x (in our example 60).
$\Pi_0 = Pr_0$;
$\Pi_t = {}_{(t-1)}p_{60} Pr_t$ for $t = 1, 2, ..., 10$.

The vector: $(\Pi_0, \Pi_1, ..., \Pi_{10})'$ is the profit signature

The profit signature is the key to assessing the profitability of the contract.

The internal rate of return (IRR) is the interest rate j such that the present value of the expected cash flows is zero. Given a profit signature $(\Pi_0, \Pi_1, ..., \Pi_n)'$ for n year contract the internal rate of return is j where:

$$\sum_{t=0}^{n} \Pi_t v_j^t = 0$$

- We can use the risk discount rate to calculate the expected present value of future profit (EPVFP), also called the net present value (NPV) of the contract. Let r be the risk discount rate.

The **NPV** is the present value, at rate r, of the projected profit signature cash flows, so that:

$$\sum_{t=0}^{n} \Pi_t v_r^t = 0$$

Profit measures

The **profit margin** is the NPV expressed as a proportion of the EPV of the premiums, evaluated at the risk discount rate.

For a contract with level premiums of P per year payable $m$thly throughout an n year contract issued to a life aged x, the profit margin is

$$Profit\ Margin = \frac{NPV}{P\ddot{a}^{(m)}_{x:\overline{n}|}}$$

Profit measures

Discounted payback period (DPP) - the break-even period.

Using the risk discount rate ,r, and is the smallest value of m such that $\sum_{t=0}^{m} \Pi_t v_r^t \geq 0$

The DPP represents the time until the insurer starts to make a profit on the contract.