Actuarial Mathematics II
MTH5125

Revision

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Plan

- Revision of life annuities and life benefits
- Single Life: Premiums and Reserves
- Death Strain at Risk and Mortality Profit
- Two lives
- Multiple States/Transitions
- Multiple Decrements
- Projected Future Cashflow Techniques

Life insurance benefits

PV:

$$
\begin{aligned}
Z &= v^{T_x} = e^{-\delta T_x} \text{ or} \\
Z &= v^{K_x+1}
\end{aligned}
$$

EPV

$$
\begin{aligned}
\bar{A}_x &= \int_0^\infty e^{-\delta t} \; {}_t p_x \; \mu_{x+t} dt \text{ or} \\
A_x &= \sum_{k=0}^{\infty} v^{k+1} \; {}_{k|} q_x
\end{aligned}
$$

Life insurance benefits

Term insurance:
PV:

$$\begin{aligned} Z &= e^{-\delta T_x} 1_{\{T_x \le n\}} \\ \text{or } Z &= v^{K_x+1} 1_{\{T_x \le n\}} \end{aligned}$$

EPV:

$$\begin{aligned} \bar{A}^{1}_{x:\overline{n}|} &= \int_0^n e^{-\delta t} \; {}_t p_x \; \mu_{x+t} dt \\ \text{or } A^{1}_{x:\overline{n}|} &= \sum_{k=0}^{n-1} v^{k+1} \; {}_{k|} q_x \end{aligned}$$

Pure endowment

PV:

$$\begin{aligned} Z &= e^{-\delta T_x} 1_{\{T_x > n\}} \\ \text{or } Z &= v^{K_x+1} 1_{\{T_x > n\}} \end{aligned}$$

EPV:

$$\bar{A}_{x:\overline{n|}}^{\;\;1} = A_{x:\overline{n|}}^{\;\;1} \equiv {}_nE_x = v^n \; {}_np_x$$

Life insurance benefits

Endowment insurance

PV:

$$\begin{aligned} Z &= v^{\min(T_x, n)} \\ \text{or } Z &= v^{\min(K_x+1, n)} \end{aligned}$$

EPV:

$$\begin{aligned} \bar{A}_{x:\overline{n}|} &= \bar{A}^{1}_{x:\overline{n}|} + v^n \; {}_n p_x \\ \text{or } A_{x:\overline{n}|} &= A^{1}_{x:\overline{n}|} + v^n \; {}_n p_x \end{aligned}$$

Life annuities

Life annuity due
PV

$$Y = 1 + v + v^2 + ...v^{K_x} = \ddot{a}_{\overline{K_x+1}|} = \frac{1 - v^{K_x+1}}{d}$$

EPV

$$\begin{aligned} \ddot{a}_x &= \frac{1 - E\left(v^{K_x+1}\right)}{d} = \frac{1 - A_x}{d} \\ &= \sum_{t=0}^{\infty} v^t \ {}_t p_x \\ &= \sum_{t=0}^{\infty} \ddot{a}_{\overline{k+1}|} \ {}_{k|}q_x \end{aligned}$$

Life annuities

Term life annuity due

PV

$$
\begin{aligned}
Y &= 1 + v + v^2 + ... v^{\min(K_x, n-1)} \\
&= \ddot{a}_{\overline{min(K_x+1,n)}|} = \frac{1 - v^{\min(K_x+1,n)}}{d}
\end{aligned}
$$

EPV

$$
\begin{aligned}
\ddot{a}_{x:\overline{n}|} &= \frac{1 - E\left(v^{\min(K_x+1,n)}\right)}{d} = \frac{1 - A_{x:\overline{n}|}}{d} \\
&= \sum_{t=0}^{n-1} v^t \;\, {}_t p_x \\
&= \sum_{t=0}^{n-1} \ddot{a}_{\overline{k+1}|} \;\, {}_{k|}q_x + \;{}_n p_x \ddot{a}_{\overline{n}|}
\end{aligned}
$$

Premiums

- Net Premium
- Gross premium: expenses are loaded into the premium

Premiums

Started with
The insurer's **random future loss** (the present value of future loss **random variable**):

$$L_0^{n/g} = PVFB_0 - PVFP_0$$

$L_0^{n/g} = PV$ of benefits outgo$(+ PV$ expenses$) - PV$ of premium income

Equation of Value/Equivalence Principle:

$$E\left(L_0^{n/g}\right) = 0$$

We can find the P (net or gross)

Premiums

The simplest possible example, whole life insurance, benefit paid at the end of the year of death, premiums paid annually in advance.
No expenses.
Net loss at issue random variable (or net random loss at issue):

$$L_0^n = Sv^{K_x+1} - P\ddot{a}_{\overline{K_x+1}|}$$

Equivalence principle/Equation of value: *Expectation of net random loss is set to zero*

$$\begin{aligned} E\left(L_0^n\right) &= 0 \\ E\left(Sv^{K_x+1}\right) &= E\left(P\ddot{a}_{\overline{K_x+1}|}\right) \\ SA_x &= P\ddot{a}_x \Leftrightarrow P = \frac{SA_x}{\ddot{a}_x} \end{aligned}$$

Premiums

The variance of the net random future loss

$$
\begin{aligned}
L_0^n &= Sv^{K_x+1} - P\ddot{a}_{\overline{K_x+1|}} \\
&= Sv^{K_x+1} - P\frac{1-v^{K_x+1}}{d} \\
&= \left(S+\frac{P}{d}\right)\left(v^{K_x+1}\right) - \frac{P}{d}
\end{aligned}
$$

$$
\begin{aligned}
V\left(L_0^n\right) &= \left(S+\frac{P}{d}\right)^2 V\left(v^{K_x+1}\right) \\
&= \left(S+\frac{P}{d}\right)^2 \left[{}^2A_x - \left(A_x\right)^2\right]
\end{aligned}
$$

Policy Values/Reserves

The insurer's **future loss at time** t:
$L_t^{n/g} = PV_t$ of future benefits ($+PV_t$ of future expenses) $-$ PV_t of future premiums
Policy value: ${}_tV = E\left(L_t^{n/g}\right)$

Policy Values/Reserves

The **gross premium policy value at time** t is the expected value (at time t) of the gross future loss random variable.

- The premiums used in the calculation are the actual premiums for the policy.

The **net premium policy value at time** t is the expected value (at time t) of the net future loss random variable.

- The premiums used in the calculation are the net premiums calculated by the Equivalence Principle, applied at the age of policy issue, calculated on the policy value basis.
- No expenses are taken into account.

Policy Values/Reserves

Recursive formula

$$\left({}_tV + P_t - e_t\right)\left(1 + i_t\right) = q_{x+t}\left(S_{t+1} + E_{t+1}\right) + p_{x+t}\ {}_{t+1}V$$

Available assets at $t+1$ = Required assets at $t+1$

Thiele's differential equation:

$$\frac{d}{dt}\ {}_tV = \delta_t\ {}_tV + \bar{P}_t - \bar{e}_t - \left(S_t + E_t - {}_tV\right)\mu_{x+t}$$

Retrospective policy values

Prospective policy values: at time t we're computing the policy value by considering what's expected to happen in the future.
Retrospective policy value at time $t =$ *accumulated value* at time t of *past premiums* - *accumulated value* at time t of *past benefits and expenses*

Retrospective policy values

Define:

$$\begin{aligned} L_{0,t} &= PV \text{ at issue of future benefits payable up to } t \\ &\quad - PV \text{ at issue of future Premiums payable up to } t \end{aligned}$$

Note that:

$$L_0 = L_{0,t} + 1\,(T_x > t)\, v^t L_t$$

Retrospective policy values

The retrospective net premium policy value is:

$$ {}_tV^R = \frac{-E\left[L_{0,t}\right](1+i)^t}{{}_tp_x} = \frac{-E\left[L_{0,t}\right]}{{}_tE_x} $$

Retrospective policy values

If:

1. the premium is calculated using equivalent principle and
2. the same basis is used for policy values, retrospective policy values and the equivalence priciple then:

$$\begin{aligned} E\left[L_0\right] &= E\left[L_{0,t} + 1\left(T_x > t\right) v^t L_t\right] = 0 \\ &\Rightarrow -E\left[L_{0,t}\right] = E\left[1\left(T_x > t\right) v^t L_t\right] \\ &\Rightarrow -E\left[L_{0,t}\right] = {}_t p_x v^t \, {}_t V^P \\ &\Rightarrow {}_t V^R = {}_t V^P \end{aligned}$$

Retrospective policy value is equal to prospective policy value.

Mortality Profit - death strain

- $$DSAR = \begin{cases} 0 & \text{if life survives to } t+1 \\ S - {}_{t+1}V & \text{if life dies in } [t, t+1) \end{cases}$$
- The maximum $S - {}_{t+1}V$ is the death strain at risk.
- The word strain is used loosely to mean a cost to the company.

Death Strain:

$$DS = S - {}_{t+1}V$$

Expected amount of the death strain is called the expected death strain (EDS).
The probability of claiming in the policy year t to $t+1$ is q_{x+t} so that:

$$EDS = q_{x+t}\left(S - {}_{t+1}V\right)$$

The actual death strain is simply the **observed** value at $t+1$ of the death strain random variable, that is:

The Mortality Profit

- The **Mortality Profit** is **Expected Death Strain – Actual Death Strain**
- Find profits for a block of policies by comparing actual experience to expected experience.
- If all lives are the same age, and subject to the same mortality table:

$$Total\ DSAR = \sum_{all\ \ policies} (S - {}_{t+1}V)$$

$$\begin{aligned} Total\ EDS &= \sum_{all\ \ policies} \left[q_{x+t} (S - {}_{t+1}V) \right] \\ &= q_{x+t} \sum_{all\ \ policies} (S - {}_{t+1}V) \\ &= q_{x+t} DSAR \end{aligned}$$

The Mortality Profit

$$\mathit{Total\ ADS} = \sum_{\mathit{all\ claims}} (S - {}_{t+1}V)$$

$$\mathit{Mortality\ Profit} = \mathit{Total\ EDS} - \mathit{Total\ ADS}$$

The joint life status

- Status that survives so long as all members are alive, and therefore fails upon the first death.
- Notation: (xy) for two lives (x) and (y)
- For two lives: $T_{xy} = \min(T_x, T_y)$

${}_tp_{xy}$ - the probability that both lives (x) and (y) survive after t years.
In the case where T_x and T_y are independent:

$$ {}_tp_{xy} = {}_tp_x \times {}_tp_y $$

${}_tq_{xy}$ is the probability that at least one of lives (x) and (y) will be dead within t years.

$$ {}_tq_{xy} = {}_tq_x + {}_tq_y - {}_tq_x \, {}_tq_y $$

Remember !!! (**even in the case of independence**):
${}_tq_{xy} \neq {}_tq_x + {}_tq_y$

The last survivor status

- Status that survives so long as there is at least one member alive, and therefore fails upon the last death.
- Notation: $(\overline{xy})$
- For two lives: $T_{\overline{xy}} = \max(T_x, T_y)$

${}_tp_{\overline{xy}}$ is the probability that at least one of lives (x) and (y) will be alive after t years.
${}_tq_{\overline{xy}}$ is the probability that both lives (x) and (y) will be dead within t years.

$$ {}_tp_{\overline{xy}} = {}_tp_x + {}_tp_y - {}_tp_{xy} = S_{T_{\overline{xy}}}(t) $$

Interpretation of Survival function

$$ S_{T_{\overline{xy}}}(t) = {}_tp_x \, {}_tp_y + {}_tp_x \, (1 - {}_tp_y) + {}_tp_y \, (1 - {}_tp_x) $$

- ${}_tp_x \, {}_tp_y$ means that both x and y alive after t years
- ${}_tp_x \, (1 - {}_tp_y)$ means that x is alive and y is dead after t years

Just as we used l_x d_x q_x from life tables for calculations involving a single life, so we also have l_{xy} d_{xy} q_{xy} which can also be written $l_{x:y}$ $d_{x:y}$ $q_{x:y}$ for extra clarity
As,

$$_tp_{xy} = \ _tp_x \ _tp_y$$

we know that

$$l_{xy} = l_x l_y$$

and

$$_tp_{xy} = \frac{l_{x+t:y+t}}{l_{x:y}}$$

and

$$d_{xy} = l_{xy} - l_{x+1:y+1}$$

so

$$q_{xy} = \frac{d_{xy}}{l_{xy}}$$

Curtate Joint Life

K_{xy} is integer part of T_{xy} - the discrete random variable which measures the curtate joint future lifetime of x and y
The probability function of K_{xy} is given by

$$\begin{aligned} P[K_{xy} &= k] = P[k \leq T_{xy} \leq k+1] \\ &= {}_{k|}q_{xy} \end{aligned}$$

Curtate Last Survivor lifetime

The curtate last survivor lifetime of (x) and y is $K_{\overline{xy}}$ - the integer part of $T_{\overline{xy}}$

$K_{\overline{xy}}$ cumulative distribution function is:

$$\begin{aligned} P[K_{\overline{xy}} &= k] = P[k \leq T_{\overline{xy}} \leq k+1] \\ &= {}_{k|}q_x + {}_{k|}q_y - {}_{k|}q_{xy} \end{aligned}$$

Whole Life Insurance discrete case

Consider an insurance under which the benefit of \$1 is paid at the end of the year of failure of status u.
Status u could be any **joint life** or **last survivor** status e.g. xy, $\overline{xy}$.

- the time at which the benefit is paid: $K_u + 1$
- the present value (at issue) of the benefit is: $Z = v^{K_u+1}$

Expected present value of benefits

$$A_u = E\left(v^{K_u+1}\right) = \sum_{k=0}^{\infty} v^{k+1} P\left[K_u = k\right] = \sum_{k=0}^{\infty} v^{k+1} \ {}_{k|}q_u$$

Variance of benefits: $Var\left(v^{K_u+1}\right) = {}^2A_u - \left(A_u\right)^2$

Whole Life Insurance continuos case

Consider an insurance under which the benefit of \$1 is paid immediately at the end (failure) of status u.
Status u could be any **joint life** or **last survivor** status e.g. xy, $\overline{xy}$.

- the time at which the benefit is paid: T_u
- the present value (at issue) of the benefit is: $Z = v^{T_u}$

Expected present value of benefits

$$\bar{A}_u = E\left(v^{T_u}\right) = \int_0^{\infty} v^t \; {}_tp_u \; \mu_{u+t} \; dt$$

Variance of benefits: $Var\left(v^{T_u}\right) = {}^2\bar{A}_u - (\bar{A}_u)^2$

Useful relations

$$A_{xy} + A_{\overline{xy}} = A_x + A_y$$

$$\bar{A}_{xy} + \bar{A}_{\overline{xy}} = \bar{A}_x + \bar{A}_y$$

Note in the discrete case:

$${}_{k|}q_{xy} = {}_{k}p_{xy} \left(1 - p_{x+k:y+k}\right) = {}_{k}p_{xy} - {}_{k+1}p_{xy}$$

Annuities benefits - discrete

Consider an n-year temporary (term) life annuity-due on status u. The present value at the issue of the benefit:

$$Y = \begin{cases} \ddot{a}_{\overline{K_u+1|}} \text{ if } K_u < n \\ \ddot{a}_{\overline{n|}} \text{ if } K_u \geq n \end{cases}$$

Annuities benefits - discrete

Expected present value of benefits for temporary annuity:

$$\ddot{a}_{u:\overline{n}|} = \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|} \ {}_{k|}q_u + \ddot{a}_{\overline{n}|} \ {}_{n}p_u$$

Remember in the case of single life:

$\ddot{a}_x = \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|} \ {}_{k|}q_u \equiv \sum_{k=0}^{n-1} v^{k+1} \ {}_{k}p_x$

Annuities benefits - discrete

$$\ddot{a}_{u:\overline{n}|} = \sum_{k=0}^{n-1} v^k \;{}_k p_u$$

or:

$$\ddot{a}_{u:\overline{n}|} = \frac{1}{d}\left(1 - A_{u:\overline{n}|}\right)$$

Variance of benefits

$$Var\left(Y\right) = \frac{1}{d^2}\left({}^2A_{u:\overline{n}|} - \left(A_{u:\overline{n}|}\right)^2\right)$$

Annuities benefits - discrete

- $\ddot{a}_{xy:\overline{n}|}$ - each payment made only if the lives (x) and (y) are alive at the time the payment is due.
- $\ddot{a}_{\overline{xy}:\overline{n}|}$ - each payment made only if at least one of the lives (x) and (y) are alive at the time the payment is due.

Consider an annuity for which the benefit of \$1 is paid each year continuously as long as a status u continues.
The present value (at issue) of the benefit: $Y = \bar{a}_{\overline{T_u}|}$
Expected present value of
benefits: $\bar{a}_u = \int_0^\infty \bar{a}_{\overline{t}|} \ {}_tp_u \ \mu_{u+t} \ dt = \int_0^\infty v^t \ {}_tp_u \ dt$
Variance of benefits: $\frac{1}{d^2}\left({}^2\bar{A}_u - (\bar{A}_u)^2\right)$

Useful Relations

Useful relations

$$\begin{aligned}\ddot{a}_{xy} + \ddot{a}_{\overline{xy}} &= \ddot{a}_x + \ddot{a}_y \\ \bar{a}_{xy} + \bar{a}_{\overline{xy}} &= \bar{a}_x + \bar{a}_y\end{aligned}$$

Premiums and Reserves

Similar to single life contracts
EPV of premiums(income)=EPV of annuity payment (outgoings)

It is possible to compute probabilities, insurances and annuities based on the failure of the status that is contingent on the order of the deaths of the members in the group, e.g. (x) dies before (y). These are called contingent functions.
Consider the probability that (x) dies before (y) - assuming independence:

$$
\begin{aligned}
P\left[T_x < T_y\right] &= \int_0^\infty f_{T_x}(t)\, S_{T_y}(t)\, dt \\
&= \int_0^\infty {}_tp_x\ \mu_{x+t}\ {}_tp_y\, dt \\
&= \int_0^\infty {}_tp_{xy}\ \mu_{x+t} dt
\end{aligned}
$$

Contingent Functions

- ${}_n q_{\overset{1}{x}y}$ is the probability that (x) dies before (y) and within n years

$$ {}_n q_{\overset{1}{x}y} = \int_0^n {}_t p_{xy} \ \mu_{x+t} dt $$

- ${}_n q_{x\overset{1}{y}}$ is the probability that (y) dies before (x) and within n years

$$ {}_n q_{x\overset{1}{y}} = \int_0^n {}_t p_{xy} \ \mu_{y+t} dt $$

Contingent Functions

Note

$$ {}_nq_{\overset{1}{x}y} + {}_nq_{x\overset{1}{y}} = 1 $$

Similarly ${}_nq_{\overset{2}{x}y}$ is the probability that (x) dies after (y) and within n years and ${}_nq_{x\overset{2}{y}}$ is the probability that (y) dies after (x) and within n years

Contingent Functions

An insurance of \$1 is payable immediately on the death of (x) provided that (y) is still alive. The present value is: 0 if $T_x > T_y$ and v^{T_x} if $T_x \leq T_y$ The expected present value of this insurance is denoted by $\bar{A}^1_{xy}$.

$$\bar{A}^1_{xy} = \int_0^\infty v^t \ {}_tp_{xy} \ \mu_{x+t} dt$$

If the benefit is payable at the end of the year of death rather than immediately, the corresponding expected present value is found by summing rather than integrating:

$$\bar{A}^1_{xy} = \sum_0^\infty v^{t+1} \ {}_tp_{xy} \ q^1_{x+t:y+t}$$

Contingent Functions

An insurance of \$1 is payable at the moment of death of (y) if predeceased by (x), i.e. if (y) dies after (x). The expected present value of this insurance is denoted by $\bar{A}_{xy}{}^{2}$. Assume (x) and (y) are independent.

$$\bar{A}_{xy}{}^{2} = \bar{A}_{y} - \bar{A}^{1}_{xy}$$

$$\bar{A}_{xy}{}^{2} = \int_{0}^{\infty} v^{t} \bar{A}_{y+t} \; {}_{t}p_{xy} \; \mu_{x+t} dt$$

Reversionary annuity

The simplest form is: an annuity that begins on the death of (x) if (y) is then alive, and continues during the remaining lifetime of (y)

- (x) is the 'counter life' or 'failing life'
- (y) is the annuitant

Notation:

- $\bar{a}_{x|y}$ - if payable continuously immediately on the death of (x), or
- $a_{x|y}$ - if payable annually in arrears from the end of the year of death of (x)

Reversionary annuity

an annuity of \$1 per year payable continuously to a life now aged y, commencing at the moment of death of (x) - briefly annuity to (y) after (x).

Simple way of calculating reversionary annuities:

$\bar{a}_{x|y} = \bar{a}_y - \bar{a}_{xy} = \frac{\bar{A}_{xy} - \bar{A}_y}{\delta}$

$a_{x|y} = a_y - a_{xy} = \frac{A_{xy} - A_y}{d}$