

Last week, we started

§ Permutations

S_n := the set of permutations
(bijections)

of $\{1, \dots, n\}$

An element f of S_n is
often written as

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n \\ f(1) & f(2) & f(3) & f(4) & \dots & f(n) \end{pmatrix}$$

We defined $f \circ g$

$$\{1, \dots, n\} \xrightarrow{g} \{1, \dots, n\}$$

$$\downarrow f$$

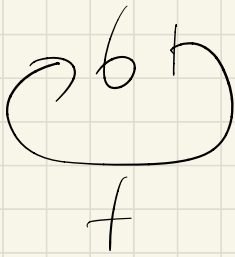
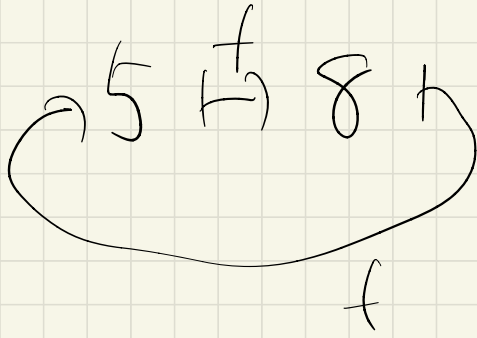
$$\{1, \dots, n\}$$

$$\S f^{-1}$$

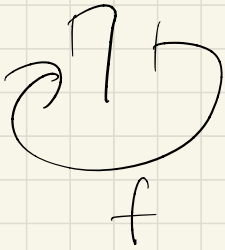
Let $n=8$

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 1 & 8 & 6 & 7 & 5 \end{pmatrix}$$

$$\Rightarrow 1 \xrightarrow{f} 2 \xrightarrow{f} 3 \xrightarrow{f} 4 \xrightarrow{f} 1$$



f is made up
of 4 different



"loops"

Def $\{\sigma_1, \dots, \sigma_N\} \subset \{1, \dots, n\}$

is a subset of N distinct

integers between 1 and n .

Write

$$(\sigma_1 \sigma_2 \dots \sigma_N)$$

to mean the permutation of $\{1, \dots, N\}$

that sends

$$\sigma_1 \mapsto \sigma_2$$

$$\sigma_2 \mapsto \sigma_3$$

:

$$\sigma_{N-1} \mapsto \sigma_N$$

$$\sigma_N \mapsto \sigma_1$$

This is called

a cycle

of length N

$(1 \leq N \leq M)$

and leaves all other integers
unchanged.

Example

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 1 & 8 & 6 & 7 & 5 \end{pmatrix}$$

$$1 \mapsto 2 \mapsto 3 \mapsto 4 \mapsto 1.$$

$$(1 \ 2 \ 3 \ 4) \in S_{\mathbb{Z}}$$

leaves 5, 6, 7, 8 unchanged

ie sends 5 to 5

6 to 6 ...

$$(6) \in S$$

is the identity map sends

$$6 \text{ to } 6$$

as well as sending

1	to	1
2	to	2
		\vdots

Similarly (17) is the identity permutation of S_8 .

The element $(\gamma_1 \gamma_2 \dots \gamma_n)$
can be written as

$$\begin{pmatrix} 1 & 2 & 3 & \dots & \delta_1 & \dots & \delta_2 & \dots & \delta_N & \dots & n \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 & 3 & \dots & \delta_2 & \delta_3 & \dots & \delta_1 & \dots & n \end{pmatrix}$$

By definition,

$$\begin{aligned} & (\delta_1 \delta_2 \dots \delta_N) \\ &= (\delta_2 \delta_3 \dots \delta_N \delta_1) \\ &= (\delta_3 \delta_4 \dots \delta_N \delta_1 \delta_2) \end{aligned}$$

Simply because they contain the same set of information.

$$\underline{\underline{PK}} \quad (\sigma_1 \sigma_2 \dots \sigma_N)^{-1} \in S_n$$

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$$(\sigma_N \sigma_{N-1} \dots \sigma_2 \sigma_1)$$

$$f = (1 \ 2 \ 3 \ 4) \in S_8$$

$$f^{-1}$$

$$1 \mapsto 4$$

$$2 \mapsto 1$$

$$3 \mapsto 2$$

$$4 \mapsto 3$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 1 & 2 & 3 & 5 & 6 & 7 & 8 \end{pmatrix}$$

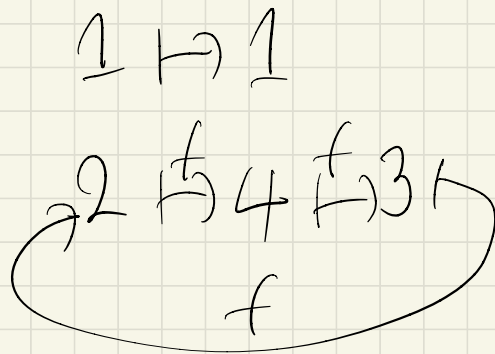
$$f^{-1} = (1 \ 4 \ 3 \ 2)$$

Example $n = 4$

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$$

Write this in the cycle form

Compute the inverse.



$$f = (243) = (432) = (324)$$

$$f^{-1} = (342) = (423) = (234)$$

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$$

$$f^{-1}: 1 \mapsto 1$$

$$2 \mapsto 3$$

$$3 \mapsto 4$$

$$4 \mapsto 2$$

$$\Rightarrow (234)$$

Def ^{If}

$$\begin{pmatrix} \sigma_1, \dots, \sigma_n \\ \sigma_1, \dots, \sigma_m \end{pmatrix} \in S_n$$

share no common element,

we say that they are disjoint

Example

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 1 & 8 & 6 & 7 & 5 \end{pmatrix}$$

$$(1234)$$

$$(58)$$

$$(6)$$

$$(7)$$

←

✓

✓

they

are 4

distinct cycles

in f .

$$f = (1234) \circ (58) \circ (6) \circ (7)$$

\uparrow

We consider them all as
elements of S_8 .

$$= (1234) \circ (58)$$

$$= (58) \circ (1234)$$

Theorem 39

Any permutation can be

written as a composition
of disjoint cycles.

This representation is unique

up to the fact that

- the cycles can be written
in any order

- each cycle can be started

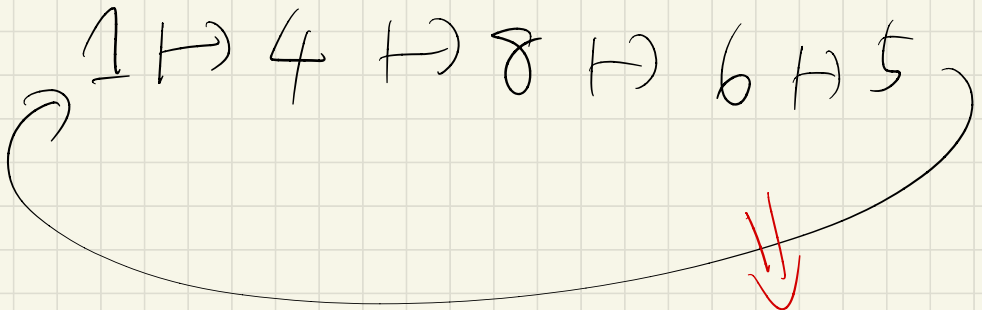
at any point

• cycles of length 1

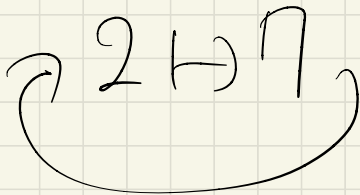
can be left out.

Example

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 7 & 3 & 8 & 1 & 5 & 2 & 6 \end{pmatrix} \in S_8$$



(14865)



\Rightarrow (27)

$$3 \mapsto 3 \Rightarrow (3)$$

$$f = (14865) \circ (27) (3)$$

$$= (14865) \circ (27)$$

$$= (27) (14865)$$

$$= (72) (86514)$$

||
⋮
}

Def Let f be an element of S_n .

The order of f is defined to

be the number of times we need to
compose f by itself to get
to identity.

To put it another way,
let $f^N = \underbrace{f \circ f \circ \dots \circ f}_{\times N}$

The order of f is the smallest N

$$\text{s.t. } f^N = 1.$$

Example

• The order of $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$ in S_4

is 2 $\equiv (13)(24)$

f f
1 \mapsto 3 \mapsto 1

2 \mapsto 4 \mapsto 2

3 \mapsto 1 \mapsto 3

4 \mapsto 2 \mapsto 4

The order of (13)
as an element of S_4

is 2.

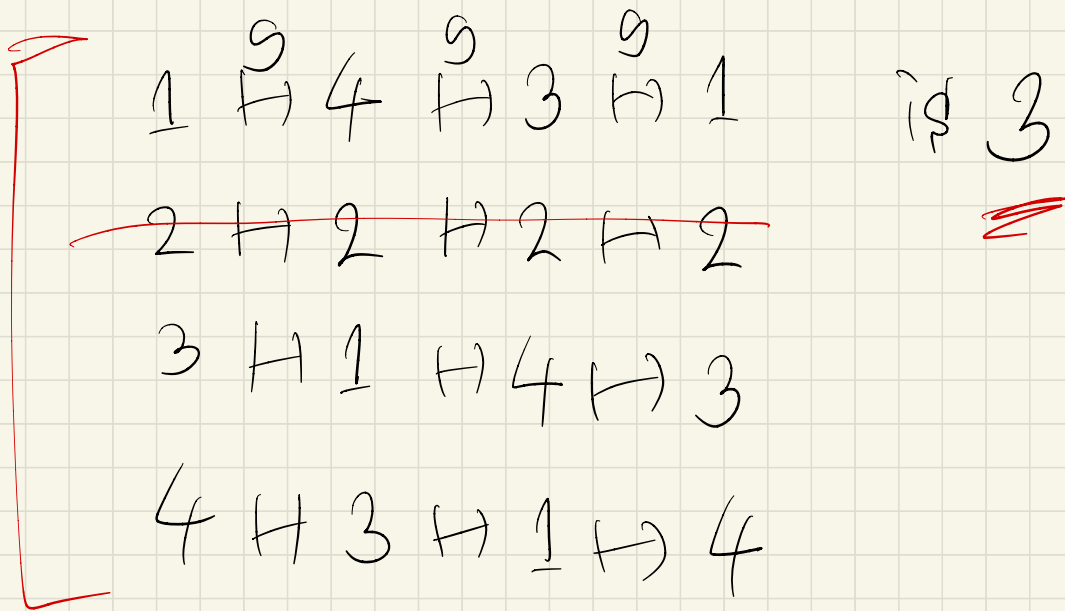
1 \mapsto 3 \mapsto 1

3 \mapsto 1 \mapsto 3

• The order of $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$

$\equiv (143)(2)$

in S_4



The order of $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$ in S_4 is 4

Prop 40

The order of a permutation is the least common multiple

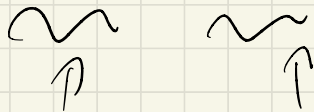
of the lengths of the cycles
/orders

in the disjoint cycle expression.

Example

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

$$= (13)(24)$$



length 2

length 2

"

"

order 2

order 2

Since lcm of 2 & 2 is 2

the order of f is 2.

Example

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 1 & 4 \end{pmatrix}$$

$$= \underbrace{(1\ 5\ 4)}_{\text{the order } 4} \circ \underbrace{(2\ 3)}_{\text{the order } 2}$$

the order
4 3

the order
2 2

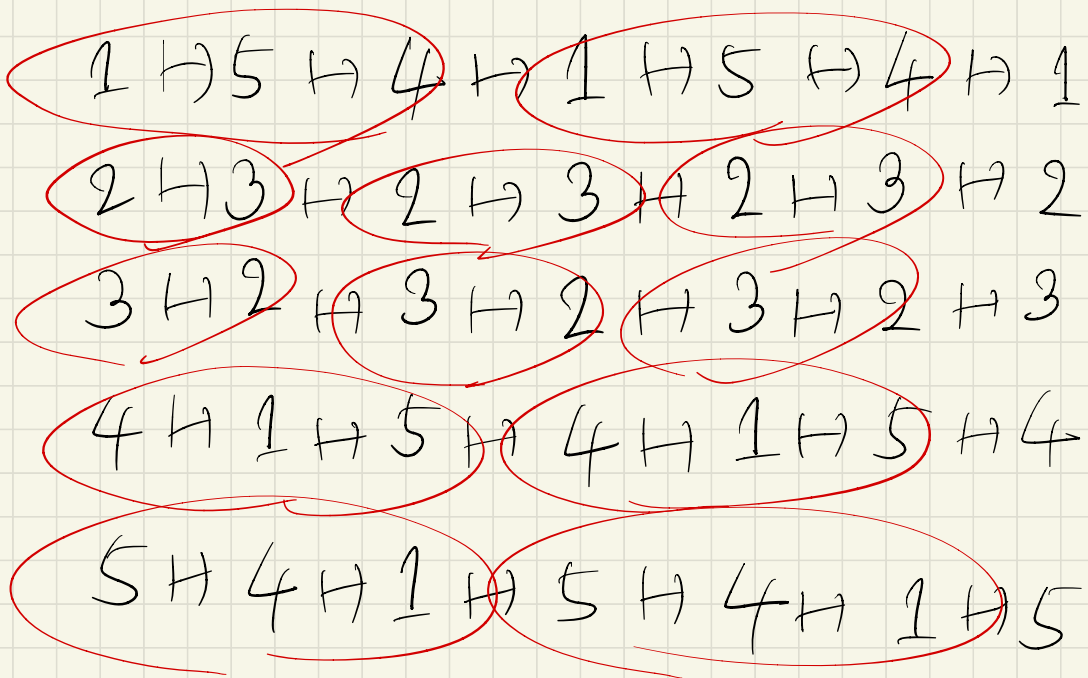
the length
3

the length
2

According to Proposition 40,

the order of f is the $\text{lcm}(3, 2)$

$$\begin{array}{l} 1\ 2\ 3\ 4\ 5 \\ 5\ 3\ 2\ 1\ 4 \end{array} \quad (1\ 5\ 4)(2\ 3) \quad = 6$$



So 6 is the order of f . \checkmark