Last week, we started

S_n Permutations

S_n := the set of permutations (bijections) of $\{1, \ldots, n\}$

An element f of S_n is often written as

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ f(1) & f(2) & f(3) & f(4) & \cdots & f(n) \end{pmatrix}$$

We defined $f \circ g$

$$\begin{tikzcd} \{1, \dots, n\} \arrow[r, "g"] & \{1, \dots, n\} \arrow[d, "f"] \\ & \{1, \dots, n\} \end{tikzcd}$$

$g \quad f^{-1},$

Let $n = 8$

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 1 & 8 & 6 & 7 & 5 \end{pmatrix}$$

$$\begin{tikzcd} 1 \arrow[r, mapsto, "f"] & 2 \arrow[r, mapsto, "f"] & 3 \arrow[r, mapsto, "f"] & 4 \arrow[lll, bend left, "f"] \end{tikzcd}$$

$$\begin{tikzcd} 5 \arrow[r, "f"] & 8 \arrow[l, bend left, "f"] \end{tikzcd}$$

$$\begin{tikzcd} 6 \arrow[loop right, "f"] \end{tikzcd}$$

$$\begin{tikzcd} 7 \arrow[loop right, "f"] \end{tikzcd}$$

f is made up of 4 different "loops"

Def $\{r_1, \dots, r_N\} \subset \{1, \dots, n\}$ is a subset of N distinct integers between 1 and n.

Write

$$(r_1 \; r_2 \; \cdots \; r_N)$$

to mean the permutation of $\{1, \cdots, n\}$ that sends

$$r_1 \mapsto r_2$$

$$r_2 \mapsto r_3$$

$$\vdots$$

$$r_{N-1} \mapsto r_N$$

$$r_N \mapsto r_1$$

This is called <u>a cycle</u> of length N

$(1 \leq N \leq n)$

and leaves all other integers unchanged.

Example

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 1 & 8 & 6 & 7 & 5 \end{pmatrix}$$

$1 \mapsto 2 \mapsto 3 \mapsto 4 \mapsto 1$.

$(1\ 2\ 3\ 4) \in S_8$

leaves $5, 6, 7, 8$ unchanged

i.e sends 5 to 5

6 to 6 ....

$(6) \in S$

is the identity map sending

6 to 6

as well as sending 1 to 1

2 to 2

$\vdots$

Similarly (7) is the identity

permutation of S_8.

The element $(\sigma_1\ \sigma_2\ \cdots\ \sigma_n)$

can be written as

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & \sigma_1 & \cdots & \sigma_2 & \cdots & \sigma_N & \cdots & n \\ 1 & 2 & 3 & & \sigma_2 & & \sigma_3 & \cdots & \sigma_1 & \cdots & n \end{pmatrix}$$

By definition,

$$(\sigma_1 \ \sigma_2 \ \cdots \ \sigma_N)$$

$$= (\sigma_2 \ \sigma_3 \ \cdots \ \sigma_N \ \sigma_1)$$

$$= (\sigma_3 \ \sigma_4 \ \cdots \ \sigma_N \ \sigma_1 \ \sigma_2)$$

simply because they contain the same set of information.

$\underline{\underline{\text{Rk}}}$ $(\sigma_1\ \sigma_2\ \cdots\cdots\ \sigma_N)^{-1} \in S_n$

$\parallel$

$(\sigma_N\ \sigma_{N-1}\ \cdots\cdots\ \sigma_2\ \sigma_1)$

$f = (1\ 2\ 3\ 4) \in S_8$

f^{-1} : $1 \mapsto 4$

$2 \mapsto 1$

$3 \mapsto 2$

$4 \mapsto 3$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 1 & 2 & 3 & 5 & 6 & 7 & 8 \end{pmatrix}$$

$f^{-1} = (1\ 4\ 3\ 2)$

Example $n = 4$

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$$

Write this in the cycle form

Compute the inverse.

$$1 \mapsto 1$$

$$\begin{tikzcd} 2 \arrow[r, mapsto, "f"] & 4 \arrow[r, mapsto, "f"] & 3 \arrow[ll, bend left, "f"] \end{tikzcd}$$

$$f = (2\,4\,3) = (4\,3\,2) = (3\,2\,4)$$

$$f^{-1} = (3\,4\,2) = (4\,2\,3) = (2\,3\,4)$$

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$$

$$f^{-1}: \begin{aligned} 1 &\mapsto 1 \\ 2 &\mapsto 3 \\ 3 &\mapsto 4 \\ 4 &\mapsto 2 \end{aligned}$$

$$\Rightarrow (2\,3\,4)$$

Def If

$$\begin{pmatrix} r_1, & \dots & r_n \\ s_1, & \dots, & s_n \end{pmatrix} \in S_n$$

share no common element,

we say that they are disjoint

Example

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 1 & 8 & 6 & 7 & 5 \end{pmatrix}$$

$(1\ 2\ 3\ 4)$

$(5\ 8)$ ←

(6) ✓

(7) ✓

they are 4 distinct cycles in f.

$$f = (1234) \circ (58) \circ (6) \circ (7)$$

We consider them all as elements of S_8.

$$= (1234) \circ (58)$$

$$= (58) \circ (1234)$$

Theorem 39

Any permutation can be written as a composition of disjoint cycles.

This representation is unique up to the fact that

- the cycles can be written in any order
- each cycle can be started

at any point

- cycles of length 1

  can be left out.

Example

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 7 & 3 & 8 & 1 & 5 & 2 & 6 \end{pmatrix} \in S_8$$

$$1 \mapsto 4 \mapsto 8 \mapsto 6 \mapsto 5$$

$$\Downarrow$$

$$(1\,4\,8\,6\,5)$$

$$2 \mapsto 7$$

$$\Rightarrow (2\,7)$$

$3 \mapsto 3 \quad \Rightarrow (3)$

$$f = (1\,4\,8\,6\,5) \circ (2\,7)(3)$$

$$= (1\,4\,8\,6\,5) \circ (2\,7)$$

$$= (2\,7)(1\,4\,8\,6\,5)$$

$$= (7\,2)(8\,6\,5\,1\,4)$$

$$\vdots$$

Def Let f be an element of S_n.

The order of f is defined to

be the number of times we need to

compose f by itself to get

the identity.

To put it another way,

let $f^N = \underbrace{f \circ f \circ \cdots \circ f}^{\times N}$

The order of f is the smallest N

s.t. $f^N = 1$.

Example

- The order of $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$ in S_4 is 2

$= (1\,3)(2\,4)$

$$\begin{array}{ccccc} & f & & f & \\ 1 & \mapsto & 3 & \mapsto & 1 \\ 2 & \mapsto & 4 & \mapsto & 2 \\ 3 & \mapsto & 1 & \mapsto & 3 \\ 4 & \mapsto & 2 & \mapsto & 4 \end{array}$$

The order of $(1\,3)$ as an element of S_4 is 2.

$1 \mapsto 3 \mapsto 1$

$3 \mapsto 1 \mapsto 3$

- The order of $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$ in S_4

$= (1\,4\,3)\cancel{(2)}$

$1 \overset{\sigma}{\mapsto} 4 \overset{\sigma}{\mapsto} 3 \overset{\sigma}{\mapsto} 1$ is 3

$\cancel{2 \mapsto 2 \mapsto 2 \mapsto 2}$

$3 \mapsto 1 \mapsto 4 \mapsto 3$

$4 \mapsto 3 \mapsto 1 \mapsto 4$

- The order of $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$ in S_4

is 4

Prop 40 The order of a permutation is the least common multiple of the lengths /orders of the cycles

in the disjoint cycle expression.

Example

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

$$= (1\,3)(2\,4)$$

length 2   length 2

"   "

order 2   order 2

Since lcm of 2 & 2 is 2

the order of f is 2.

Example

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 1 & 4 \end{pmatrix}$$

$$= \underbrace{(1\ 5\ 4)}\circ\underbrace{(2\ 3)}$$

the order is 3   the order is 2

the length 3   the length 2

According to Proposition 40,

the order of f is the $\text{lcm}(3,2)$

$= 6$

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 1 & 4 \end{matrix}$$

$(154)(23)$

$1 \mapsto 5 \mapsto 4 \mapsto 1 \mapsto 5 \mapsto 4 \mapsto 1$

$2 \mapsto 3 \mapsto 2 \mapsto 3 \mapsto 2 \mapsto 3 \mapsto 2$

$3 \mapsto 2 \mapsto 3 \mapsto 2 \mapsto 3 \mapsto 2 \mapsto 3$

$4 \mapsto 1 \mapsto 5 \mapsto 4 \mapsto 1 \mapsto 5 \mapsto 4$

$5 \mapsto 4 \mapsto 1 \mapsto 5 \mapsto 4 \mapsto 1 \mapsto 5$

So 6 is the order of f.