MTH5114 Linear Programming and Game Theory, Spring 2024

Week 8 Coursework Questions

Viresh Patel

These exercises should be completed individually and submitted (together with those of weeks 9 and 10) via the course QMPlus page by **9am on Tuesday, 09 April**.

Make sure you clearly write your **name** and **student ID** number at the top of your submission:.

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1. Give the dual of each of the following linear programs:

   (a)

$$\begin{aligned} \text{maximize} \quad & 100x_1 + 200x_2 + 300x_3 + 400x_4 \\ \text{subject to} \quad & x_1 + 2x_2 + 3x_3 - 4x_4 \leq 10, \\ & -5x_1 - 6x_2 - 7x_3 + 8x_4 \geq -20, \\ & 9x_1 + 10x_2 + 11x_3 - 12x_4 = 30, \\ & x_1, x_3 \geq 0, \\ & x_4 \leq 0, \\ & x_2 \text{ unrestricted} \end{aligned}$$

Solution: We first fix the constraints and sign restrictions

$$\begin{aligned} \text{maximize} \quad & 100x_1 + 200x_2 + 300x_3 - 400x_4 \\ \text{subject to} \quad & x_1 + 2x_2 + 3x_3 + 4\bar{x}_4 \leq 10, \\ & 5x_1 + 6x_2 + 7x_3 + 8\bar{x}_4 \leq 20, \\ & 9x_1 + 10x_2 + 11x_3 + 12\bar{x}_4 = 30, \\ & x_1, x_3 \geq 0, \\ & \bar{x}_4 \geq 0, \\ & x_2 \text{ unrestricted} \end{aligned}$$

Then we can take the dual.

$$\begin{aligned} \text{minimize} \quad & 10y_1 + 20y_2 + 30y_3 \\ \text{subject to} \quad & 1y_1 + 5y_2 + 9y_3 \geq 100, \\ & 2y_1 + 6y_2 + 10y_3 = 200, \\ & 3y_1 + 7y_2 + 11y_3 \geq 300, \\ & 4y_1 + 8y_2 + 12y_3 \geq -400, \\ & y_1, y_2 \geq 0, \\ & y_3 \text{ unrestricted} \end{aligned}$$

(b)

$$\begin{aligned} \text{maximize} \quad & x_4 \\ \text{subject to} \quad & x_4 \leq x_1 - 2x_2 + 4x_3, \\ & x_4 \leq -2x_1 + 3x_2 - 5x_3, \\ & x_4 \leq 3x_1 - 4x_2 + 6x_3, \\ & x_1 + x_2 + x_3 = 1, \\ & x_1, x_2, x_3 \geq 0, \\ & x_4 \text{ unrestricted} \end{aligned}$$

Hint: Remember to first rewrite all the constraints so that all variables are on the left and the right-hand side is a constant.

Solution: We need to rearrange our inequalities first, to put the constant term on the RHS:

$$\begin{aligned} \text{maximize} \quad & x_4 \\ \text{subject to} \quad & -x_1 + 2x_2 - 4x_3 + x_4 \leq 0, \\ & 2x_1 - 3x_2 + 5x_3 + x_4 \leq 0, \\ & -3x_1 + 4x_2 - 6x_3 + x_4 \leq 0, \\ & x_1 + x_2 + x_3 = 1, \\ & x_1, x_2, x_3 \geq 0, \\ & x_4 \text{ unrestricted} \end{aligned}$$

Then, taking the dual gives:

$$\begin{aligned} \text{minimize} \quad & y_4 \\ \text{subject to} \quad & -y_1 + 2y_2 - 3y_3 + y_4 \geq 0, \\ & 2y_1 - 3y_2 + 4y_3 + y_4 \geq 0, \\ & -4y_1 + 5y_2 - 6y_3 + y_4 \geq 0, \\ & y_1 + y_2 + y_3 = 1, \\ & y_1, y_2, y_3 \geq 0, \\ & y_4 \text{ unrestricted} \end{aligned}$$

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Week 9 Coursework Questions — Viresh Patel

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1. For the following linear program,

$$\begin{aligned} \text{maximize} \quad & 7x_1 + 12x_2 + 9x_3 \\ \text{subject to} \quad & x_1 + 3x_2 + 3x_3 \leq 4, \\ & 2x_1 + 3x_2 + 2x_3 \leq 5, \\ & 2x_1 + 4x_2 + 3x_3 \leq 7, \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

determine whether $\mathbf{x}^\mathsf{T} = (\frac{3}{2}, \frac{1}{3}, \frac{1}{2})$ is an optimal solution using the principle of complementary slackness.

Solution: We first compute the dual.

$$\begin{aligned} \text{minimize} \quad & 4y_1 + 5y_2 + 7y_3 \\ \text{subject to} \quad & y_1 + 2y_2 + 2y_3 \geq 7, \\ & 3y_1 + 3y_2 + 4y_3 \geq 12, \\ & 3y_1 + 2y_2 + 3y_3 \geq 9, \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

Then we check that $\mathbf{x}$ is feasible (checking which constraints are tight and which variables are non-zero). We find that $\mathbf{x}$ is feasible and find that first and second primal constraints are tight, but the third is not, and that all primal variables are ≥ 0. Since $x_1, x_2, x_3 \geq 0$, so any $\mathbf{y}$ that satisfies complementary slackness, together with $\mathbf{x}$ must make all dual constraints tight:

$$\begin{aligned} y_1 + 2y_2 + 2y_3 &= 7 \\ 3y_1 + 2y_2 + 3y_3 &= 9 \\ 3y_1 + 3y_2 + 4y_3 &= 12 \end{aligned}$$

Additionally, since the third primal constraint is not tight, any $\mathbf{y}$ that satisfies complementary slackness together with $\mathbf{x}$ must have:

$$y_3 = 0$$

Solving these equations, we obtain a unique solution $\mathbf{y}^\mathsf{T} = (1, 3, 0)$. Clearly $\mathbf{y} \geq \mathbf{0}$, and $\mathbf{y}$ satisfies all dual constraints by construction. Thus, $\mathbf{x}$ is an optimal solution to the program and $\mathbf{y}^\mathsf{T} = (1, 3, 0)$ is an optimal solution to the dual of the program.

2. For the following linear program

$$\begin{aligned} \text{maximize} \quad & x_1 + 8x_2 + 3x_3 \\ \text{subject to} \quad & 2x_1 + 8x_2 + 2x_3 \leq 4, \\ & 2x_1 + 4x_2 + 3x_3 \leq 4, \\ & x_1 + 2x_2 + x_3 \leq 1, \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

determine whether $\mathbf{x}^\mathsf{T} = (0, \frac{1}{2}, 0)$ is an optimal solution using the principle of complementary slackness.

Solution: First we compute the dual:

$$\begin{aligned} \text{minimize} \quad & 4y_1 + 4y_2 + y_3 \\ \text{subject to} \quad & 2y_1 + 2y_2 + y_3 \geq 1, \\ & 8y_1 + 4y_2 + 2y_3 \geq 8, \\ & 2y_1 + 3y_2 + y_3 \geq 3, \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

We find that the given $\mathbf{x}$ is feasible for the primal LP with only the second primal constraint being not tight. Since $x_2 > 0$ then any $\mathbf{y}$ that satisfies complementary slackness together with $\mathbf{x}$ must make the second dual constraint tight:

$$8y_1 + 4y_2 + 2y_3 = 8$$

Additionally, since the second primal constraint is not tight, any $\mathbf{y}$ that satisfies complementary slackness together with $\mathbf{x}$ must have:

$$y_2 = 0$$

Here, we only have 1 equation for 3 variables. Any solution with $y_2 = 0$, and $4y_1 + y_3 = 4$ will satisfy the complementary slackness conditions (and also the 2nd constraint of the dual). We need to see if we can find some solution that satisfies this equation, has $y_1, y_2, y_3 \geq 0$ and also satisfies the other 2 dual constraints, which (since $y_2 = 0$) can be simplified to:

$$\begin{aligned} 2y_1 + y_3 &\geq 3 \\ 2y_1 + y_3 &\geq 1 \end{aligned}$$

One way to do this is to set $y_1 = \frac{1}{4}$, $y_2 = 0$ and $y_3 = 3$. Then, clearly $y_1, y_2, y_3 \geq 0$ and also $4y_1 + y_3 = 1 + 3 = 4$, and:

$$2 \cdot \frac{1}{4} + 3 = \frac{1}{2} + 3 > 3 > 1$$

Thus, both remaining dual constraints will be satisfied. It follows that $\mathbf{x}$ is optimal, and $\mathbf{y}^\mathsf{T} = (\frac{1}{4}, 0, 3)$ is an optimal solution to the dual. In fact, one can show that in this case any solution: $\mathbf{y}^\mathsf{T} = (\alpha, 0, 4 - 4\alpha)$ with $0 \leq \alpha \leq \frac{1}{2}$ is optimal for the dual.

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Week 10 Coursework Questions

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1. Consider the following 2 player, zero-sum game. The row and column player each have 3 cards labelled 1, 2, and 3. Both players select a card, and then they simultaneously reveal their selected cards. The player with the highest numbered card wins. The loser pays the winner an amount equal to the number on the winner's card. If the number on both players' cards is the same, neither player wins or loses anything.

   Give the payoff matrix for this game (from the perspective of the row player) and identify any Nash equilibria that the game has.

   Solution:

	1	2	3
1	0	-2	-3
2	2	0	-3
3	3	3	0

   The outcome $(3, 3)$, i.e. where both player play the card 3, is the only Nash equilibrium for this game

2. Give an example of a 2-player, zero-sum game with the following properties (by giving its payoff matrix from the perspective of the row player):

   - The row player has strategy set $\{r_1, r_2\}$ and the column player has strategy set $\{c_1, c_2\}$
   - The security levels of r_1, r_2, c_1, and c_2 are respectively -1, -2, 4, and 3.

   Solution: The only possibilities are

	c_1	c_2
r_1	-1	3
r_2	4	-2

   and

	c_1	c_2
r_1	4	-1
r_2	-2	3