Assignment 5
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Notes for attempting Assignment 5

WMT = Weierstrass M-Test

See assignment 5 for precise wording of questions

Q1 Is $f_n(x) = \dfrac{1+\cos^{2023}(nx)}{\sqrt{n}}$ UC to a function $f: \mathbb{R} \to \mathbb{R}$

Note: $\begin{cases} |\cos(nx)| \leq 1 \quad \forall n, \ \forall x \in \mathbb{R} \\ |\cos(nx)|^k \leq 1, \quad k \in \mathbb{N}. \end{cases}$

$$|f_n(x)| = \frac{|1+\cos^{2023}(x)|}{\sqrt{n}} \leq \frac{|1|+|\cos^k(\ )|}{\sqrt{n}} \leq \frac{2}{\sqrt{n}}$$

$\therefore |f_n(x)| \to 0$ as $n \to 0$ $\because \frac{2}{\sqrt{n}} \to 0$. $\therefore f(x) = 0$

Also by WMT $f_n(x)$ converges uniformly to $f(x) = 0$.

Q2

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos\left(\frac{x}{n}\right) \qquad \text{Is } f_k(x) = \sum_{n=1}^{k} \frac{1}{n^2} \cos\left(\frac{x}{n}\right) \text{ UC to } f(x)?$$

$$\sum_{n=1}^{\infty} \left|\frac{1}{n^2} \cos\left(\frac{x}{n}\right)\right| \leq \sum_{n=1}^{\infty} \frac{1}{n^2}, \ \forall x \in \mathbb{R}$$

$$\left(\sum_{n=1}^{\infty} \frac{1}{n^a} \text{ is C: for } a > 1 \text{ ; D: for } a \leq 1\right)$$

$$\therefore \sum_{n=1}^{k} \frac{1}{n^2} \cos\left(\frac{x}{n}\right) \text{ is uniformly convergent (by WMT)}$$

to the limit function $f(x) = \frac{1}{n^2}\sum_{n=1}^{\infty} \frac{1}{n^2} \cos\left(\frac{x}{n}\right)$ as $k \to \infty$

$\sum_{n=1}^{\infty} \frac{1}{n^3} \sin\left(\frac{x}{n}\right)$ - same argument as above using WMT

$$\text{for } \sum_{n=1}^{\infty} \left|\frac{1}{n^3} \sin\left(\frac{x}{n}\right)\right| \leq \sum_{n=1}^{\infty} \frac{1}{n^3}$$

Q3 Does $f(x)=\sum_{n=1}^{\infty} \frac{1}{n}\cos\left(\frac{x}{n}\right)$ exist as a function?

WMT?

Well, the obvious comparison is (as in Q2)

$$\left|\frac{1}{n}\cos\left(\frac{x}{n}\right)\right| \leq \left|\frac{1}{n}\right|$$

$$\sum_{n=1}^{\infty}\left|\frac{1}{n}\cos\left(\frac{x}{n}\right)\right| \leq \sum_{n=1}^{\infty}\frac{1}{n} \quad \text{(is not convergent)}$$

WMT does not deliver uniform conv using the comparison with $\sum \frac{1}{n}$ $\because$ is a divergent series.

$$\left|\frac{1}{n}\cos\left(\frac{x}{n}\right)\right| \leq \left|\frac{1}{n}\right| \quad \forall\, x \in \mathbb{R}$$

However, for $x=0$, $\left|\frac{1}{n}\cos\left(\frac{x}{n}\right)\right| = \frac{1}{n}$, and so

$$f(0) = \sum_{n=1}^{\infty}\frac{1}{n}\cos\left(\frac{x}{n}\right)\Bigg|_{x=0} = \sum_{n=1}^{\infty}\frac{1}{n} \to \infty \text{ (divergent)}$$

i.e. $f(0)$ is not defined and so a pointwise limit-function $f(x)$ does not exist for all $x \in \mathbb{R}$ (fails for $x=0$).

NO FURTHER QUESTIONS ON f can be addressed!

Q4

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos\left(\frac{x}{n}\right)$$ – pointwise limit function of

$$f_k(x) = \sum_{n=1}^{k} \frac{1}{k^2} \cos\left(\frac{x}{k}\right)$$ converges uniformly to $f(x)$

by the WMT see Q2.

$$g(x) = \sum_{n=1}^{\infty} \frac{1}{n^3} \sin\left(\frac{x}{n}\right)$$ – pointwise limit function of

$$g_k(x) = \sum_{n=1}^{k} \frac{1}{k^2} \sin\left(\frac{x}{k}\right)$$ converges uniformly to $g(x)$

<u>Relationship between</u> $\frac{1}{n^2}\cos\left(\frac{x}{n}\right)$ & $\frac{1}{n^3}\sin\left(\frac{x}{n}\right)$?

$$\frac{d}{dx}\left(\frac{1}{n^2}\cos\left(\frac{x}{n}\right)\right) = \frac{1}{n^2}\left(-\sin\left(\frac{x}{n}\right)\right)\frac{1}{n}$$

$$= \frac{1}{n^3}\left(-\sin\left(\frac{x}{n}\right)\right) = -\frac{1}{n^3}\sin\left(\frac{x}{n}\right)$$

$$\lim_{k\to\infty} f_k(x) = \lim_{k\to\infty} \sum_{n=1}^{k} \frac{1}{n^2}\cos\left(\frac{x}{n}\right) = f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2}\cos\left(\frac{x}{n}\right)$$

$$\lim_{k\to\infty} g_k(x) = \lim_{k\to\infty} -\sum_{n=1}^{k} \frac{1}{n^3}\sin\left(\frac{x}{n}\right) = g(x) = -\sum_{n=1}^{\infty} \frac{1}{n^3}\sin\left(\frac{x}{n}\right)$$

The partial sums $f_k(x)$, $g_k(x)$ are finite sums of cont fns (cos / sin) and therefore continuous.

<u>THM</u> UC of a sequence of continuous fns is continuous.

$\therefore$ $f(x)$, $g(x)$ are continuous.

Unfortunately, the Q4 asks to prove that f is differentiable.

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n^2}\cos\frac{x}{n} = \sum_{n=0}^{\infty}\int^{x} -\frac{1}{n^3}\sin\left(\frac{t}{n}\right)dt = -\int^{x}\sum_{n=0}^{\infty}\frac{1}{n^3}\sin\left(\frac{t}{n}\right)dt = -\int^{x} g(t)\,dt$$

and $g(t)$ is continuous. The interchange of $\int$ and Σ can be made because of UC $\therefore$ By the Fund Thm of Calc. $f'(x) = -g(x)$

$\therefore$ $f(x)$ is differentiable