Week 12

Plan: (1) The quadrupole formula

(2) The chirp (mass)

At leading order in the small velocity (Post-Newtonian) limit we saw that a binary emits a GW whose frequency Ω_{GW} is twice the orbital frequency. We can explicitly extract the two polarisations by choosing a basis for the space orthogonal to $\hat{n}$ (see pag. 7 Week 7)

$$\hat{n} = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta).$$

The vectors $\frac{\partial \hat{n}}{\partial \vartheta}$ and $\frac{\partial \hat{n}}{\partial \varphi}$ are orthogonal to $\hat{n}$ (since $\hat{n}^2 = 1 \Rightarrow \frac{\partial \hat{n}^2}{\partial \vartheta} = 0 \Rightarrow \hat{n} \cdot \frac{\partial \hat{n}}{\partial \vartheta} = 0$ and similarly for $\frac{\partial \hat{n}}{\partial \varphi}$)

$$\frac{\partial \hat{n}}{\partial \vartheta} = (\cos\vartheta\cos\varphi, \cos\vartheta\sin\varphi, -\sin\vartheta) \equiv e_\vartheta$$

$$\frac{\partial \hat{n}}{\partial \varphi} = \sin\vartheta(-\sin\varphi, \cos\varphi, 0) \equiv \sin\vartheta\, e_\varphi$$

Thus we have $\begin{cases} e_\vartheta e_\vartheta = e_\varphi e_\varphi = 1, \quad e_\vartheta e_\varphi = 0 \\ e_\vartheta \hat{n} = e_\varphi \hat{n} = 0 \end{cases}$

For $\vartheta = \varphi = 0$ we are in the situation of pag. 5 Week 11

with $\hat{n} = (0,0,1)$, $e_\vartheta = (1,0,0)$, $e_\varphi = (0,1,0)$.

Then $\frac{1}{\sqrt{2}} \left[(e_\vartheta)^i (e_\varphi)^j + (e_\vartheta)^j (e_\varphi)^i \right] H_{ij} \equiv H_\times$ and

$\frac{1}{\sqrt{2}} \left[(e_\vartheta)^i (e_\vartheta)^j H_{ij} - (e_\varphi)^i (e_\varphi)^j H_{ij} \right] \equiv H_+$

The first step generalises the definition of $H_{\times,+}$ for any value of ϑ, φ. Thus by using

$$U_{\langle ij \rangle} = -\mu \frac{a^2}{2} \Omega_{GW}^2 \begin{pmatrix} \cos(\Omega_{GW} u) & \sin(\Omega_{GW} u) & 0 \\ \sin(\Omega_{GW} u) & -\cos(\Omega_{GW} u) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

we have

$$h_\times \simeq \frac{2G}{r} (e_\vartheta)^k (e_\varphi)^l \, U_{\langle kl \rangle} =$$

$$\frac{2G}{r} \sqrt{2} \cos\vartheta \sin(2\varphi - 2\Omega_{orb} u) (-2\mu a^2 \Omega_{orb}^2)$$

$$h_+ \simeq \frac{2G}{r} \frac{e_\vartheta^k e_\vartheta^l - e_\varphi^k e_\varphi^l}{\sqrt{2}} U_{\langle kl \rangle} =$$

$$-\frac{2G}{r} \frac{1+\cos^2\vartheta}{\sqrt{2}} \cos(2\varphi - 2\Omega_{orb} u) (-2\mu a^2 \Omega_{orb}^2)$$

where I used $\Omega_{GW} = 2\Omega_{orb}$

Let us now use the result above to calculate how much energy is radiated away through GW. In order to do so we need to go beyond the linearised analysis as we expect E_{rad} depends quadratically on h. See section 7.2.4 of the LATEX notes for a derivation where Einstein's equation are expanded up to second order in h leading to

$d\Omega_{\vartheta\varphi} = d\vartheta \sin\vartheta \, d\varphi$

Total radiated energy $\rightarrow$

$$\boxed{E_{rad} = \frac{1}{32\pi G} \int dt \, d\Omega_{\vartheta\varphi} \sum_{a=\times,+} \left| \frac{d(r\bar{h}_a)}{dt} \right|^2}$$

The factors of r cancel in $r\bar{h}$! In order to properly define this result, let us calculate the radiated energy in one period of orbital motion, i.e. we restrict the integral over t to $\left[0, \frac{2\pi}{\Omega_{orb}} \equiv T_{orb}\right]$. We have

$$E_{rad}^{1\text{-period}} = \frac{1}{16\pi G} \int_0^{T_{orb}} dt \left(4G\mu a^2 \Omega_{orb}^2\right)^2 \int d\Omega_{\vartheta\varphi} \Big\{ \left[\partial_t\left(\cos\vartheta \sin\left(2\varphi - 2\Omega_{orb} u\right)\right)\right]^2$$

$$\left[\partial_t\left(\frac{1+\cos^2\vartheta}{2} \cos\left(2\varphi - 2\Omega_{orb} u\right)\right)\right]^2 \Big\}$$

We have $\displaystyle\int_0^{T_{orb}} dt \sin^2\left(2\varphi - 2\Omega_{orb} u\right) = \int_0^{T_{orb}} dt \sin^2\left(2\varphi - 2\Omega_{orb} u\right) = \frac{T_{orb}}{2}$

so

$$E_{rad}^{1\text{-period}} = \frac{1}{16\pi G} G^2 \mu^2 \left(4 a^2 \Omega_{orb}^2\right)^2 \frac{T_{orb}}{2} 4\Omega_{orb}^2$$

$$\int d\Omega_{\vartheta\varphi} \left[\cos^2\vartheta + \frac{1+2\cos^2\vartheta + \cos^4\vartheta}{4} \right]$$

By focusing on the second line we have

$$\int_0^\pi d\vartheta \sin\vartheta \left(\int_0^{2\pi} d\varphi\right) \frac{1}{4}\left[1 + 6\cos^2\vartheta + \cos^4\vartheta\right] =$$

$$-\frac{\pi}{2}\int_0^\pi d(\cos\vartheta)\left[1+6\cos^2\vartheta+\cos^4\vartheta\right] = \frac{\pi}{2}\left[\cos\vartheta + 2\cos^3\vartheta + \frac{\cos^5\vartheta}{5}\right]_\pi^0 =$$

$$\pi\left(1+2+\frac{1}{5}\right) = \frac{16\pi}{5}$$

Thus the flux of energy in 1-period $\langle P \rangle = \frac{E^{1rad}}{T_{orb}}$ is

$$\langle P \rangle = \frac{1}{16\pi G} G^2 \mu^2 \, 64 \, a^4 \Omega_{orb}^6 \frac{1}{2} \frac{16\pi}{5}$$

By using $a^3 \Omega_{orb}^2 = GM$ we have

$$\langle P \rangle = \frac{32 G^4 \mu^2 M^3}{5 a^5} = \frac{32}{5} G^4 \mu^2 M^3 \left(\frac{\Omega_{orb}^2}{GM}\right)^{5/3} =$$

$$= \frac{32}{5} \frac{\nu^2}{G} \underbrace{\left(GM\Omega_{orb}\right)}_{v^3}^{10/3} \qquad \nu \equiv \frac{\mu}{M} = \frac{m_1 m_2}{(m_1+m_2)^2}$$

v^3 using $\Omega_{orb} = \frac{v}{a}$ and $a^3\Omega_{orb}^2 = GM$

A small $\left(\sim\left(\frac{v}{c}\right)^{10}\right)$ amount of energy is radiated in each period... where does it come from? It has to come from a variation of the mechanical energy of the system $|E| = \frac{1}{2}\frac{GM\mu}{a} = \frac{1}{2}GM\mu\left(\frac{\Omega_{orb}^2}{GM}\right)^{1/3} = \frac{\mu}{2}\left(GM\Omega_{orb}\right)^{2/3}$

Thus we can write a balance equation

$$\boxed{\frac{d}{dt}|E| = \langle P \rangle}$$

which implies that the orbit radius a, or equivalently the orbit period Ω_{orb}, must change over time. Thus we should introduce $a(t)$ and $\Omega_{orb}(t)$ which are slowly varying functions (with respect to Ω_{orb} itself)

$$\frac{d|E|}{dt} = -\frac{1}{2}\frac{GM\mu}{a^2(t)}\frac{da}{dt} = \frac{32\,G^4\mu^2M^3}{5\,a^5(t)} \Rightarrow$$

$$a^3(t)\,da(t) = -\frac{64}{5}G^3\mu M^2 dt \Rightarrow a^4(t) - a_i^4 = -\frac{256}{5}G^3\mu M^2 t$$

Here a_i is the radius at $t=0$ and we see that $a(t)$ gets smaller as t grows till (formally) vanishes at

$$t_{merger} = \frac{5}{256}\frac{a_i^4}{G^3 m_1 m_2 (m_1+m_2)}$$

which we can interpret as the time of the merger!

Similarly in terms of $\Omega_{orb}(t)$

$$\frac{\mu}{2}\frac{d}{dt}\left(GM\Omega_{orb}(t)\right)^{2/3} = \frac{32}{5}\frac{\nu^2}{G}\left(GM\Omega_{orb}\right)^{10/3} \Rightarrow$$

$$\frac{\mu}{3}(GM)^{2/3}\Omega_{orb}^{-1/3}\frac{d\Omega_{orb}}{dt} = \frac{32}{5}\frac{\nu\mu}{GM}\left(GM\Omega_{orb}\right)^{10/3}$$

$$\frac{1}{\Omega_{orb}}\frac{d\Omega_{orb}}{dt} = \frac{96}{5}\frac{\nu}{GM}\left(GM\Omega_{or}\right)^{8/3}$$

Thus plotting the variation of Ω_{orb} we get info about a particular combination of the masses M_{chirp}

$$d\Omega_{orb} = \frac{96}{5}\Big(\underbrace{GM\nu^{3/5}}_{\equiv GM_{chirp}}\Big)^{5/3}\Omega_{orb}^{11/3}\,dt$$

$$\boxed{M_{chirp} = (m_1+m_2)\left(\frac{m_1 m_2}{(m_1+m_2)^2}\right)^{3/5} = \frac{m_1^{3/5}\,m_2^{3/5}}{(m_1+m_2)^{1/5}}}$$

This is how we extract a first data point to determine the masses of the inspiralling bodies!