Week 11

- Assessed coursework 2

deadline 11am

(15th of April)

to be submitted via *QMPLUS*

- A Mock exam paper

$\boxed{\text{Typo}}$ $\cdots = \gcd(x^8 + [2],$

$\to [2]x^6 + [2])$

§ Permutations

Def Let S be a set.

A permutation of S is a function $f : S \to S$ which is bijection.

injective & surjective

injective $=$ if $f(r) = f(s)$ $r, s \in S$ then $r = s$

surjective $=$ given $s \in S$, there exists $r \in S$ s.t. $f(r) = s$.

Fix $n \geq 1$.

<u>Def</u> The set of permutations of

$$S = \{1, \cdots, n\}$$

is denoted by S_n.

and every element of S_n is written

as

$$f = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ f(1) & f(2) & f(3) & \cdots & f(n) \end{pmatrix}$$

<u>Rk</u> Since f is bijective,

$$\{1, \cdots, n\} = \{f(1), \cdots, f(n)\}$$

Example $n=8$

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 7 & 3 & 8 & 1 & 5 & 2 & 6 \end{pmatrix}$$

↑ the second row is a reordering of $1, \cdots, 8$.

f is a bijection

$$\{1, \dots, 8\} \to \{1, \dots 8\}$$

sending

$$\begin{aligned} 1 &\mapsto 4 \\ 2 &\mapsto 7 \\ 3 &\mapsto 3 \\ 4 &\mapsto 8 \\ 5 &\mapsto 1 \\ 6 &\mapsto 5 \\ 7 &\mapsto 2 \\ 8 &\mapsto 6 \end{aligned}$$

Rk It's rare to see otherwise but the first row does NOT have to be in the order 1, 2, 3, ...

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 7 & 3 & 8 & 1 & 5 & 2 & 6 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 5 & 1 & 8 & 6 & 7 & 4 \\ 7 & 3 & 1 & 4 & 6 & 5 & 2 & 8 \end{pmatrix}$$

because they contain the same set of information.

Prop36

$$|S_n| = n!$$

Pf Follows by definition

<u>Rk</u> $S_n = \mathrm{Sym}(\{1, \dots, n\})$

<u>Def</u> If f and g are permutations of S, then define $f \circ g$ to be the composition

$$\begin{array}{ccccc} S & \xrightarrow{g} & S & \xrightarrow{f} & S \\ \in & & \in & & \in \\ s & \mapsto & g(s) & \mapsto & f(g(s)) \end{array}$$

i.e. $(f \circ g)(s) = f(\, g(s) \,)$

Prop 3.7 If f and g are elements of S_n

$f, g : \{1, \cdots, n\} \to \{1, \cdots, n\}$

then $f \circ g$ is also an element in S_n.

Pf We need to check that

$f \circ g$ is bijective on $\{1, \cdots, n\}$

injectivity Need to check:

$x, y \in \{1, \cdots, n\}$

if $(f \circ g)(x) = (f \circ g)(y)$

then $x = y$

To do this, assume

$$(f \circ g)(x) = (f \circ g)(y).$$

$$\shortparallel \;\leftarrow \text{definition} \rightarrow\; \shortparallel$$

$$f(\,g(x)\,) \qquad\qquad f(\,g(y)\,)$$

Since f <u>is</u> injective.

$$g(x) = g(y)$$

Since g is injective,

$$x = y. \quad \square$$

<u>Surjectivity</u> Need to check:

given $z \in \{1, \cdots, n\}$,

there exists $x \in \{1, \cdots, n\}$

s.t. $(f \circ g)(x) = z$.

<u>Example</u> $n = 8$

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 7 & 3 & 8 & 1 & 5 & 2 & 6 \end{pmatrix}$$

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 3 & 1 & 8 & 7 & 2 & 5 & 4 \end{pmatrix}$$

$f \circ g = ?$

$$
\begin{array}{ccccc}
1 & \overset{g}{\mapsto} & 6 & \overset{f}{\mapsto} & 5 \\
2 & \mapsto & 3 & \mapsto & 3 \\
3 & \mapsto & 1 & \mapsto & 4 \\
4 & \mapsto & 8 & \mapsto & 6 \\
5 & \mapsto & 7 & \mapsto & 2 \\
6 & \mapsto & 2 & \mapsto & 7 \\
7 & \mapsto & 5 & \mapsto & 1 \\
8 & \mapsto & 4 & \mapsto & 8
\end{array}
$$

$$
f \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 3 & 4 & 6 & 2 & 7 & 1 & 8 \end{pmatrix}
$$

<u>Prop 38</u> If f is an element of S_n, then the inverse f^{-1} exists

in S_n

Rk f^{-1} is the function g

s.t. $f \circ g = g \circ f = 1$

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 3 & \cdots & n \end{pmatrix}$$

"Proof" Define $g : \{1, \cdots, n\} \to \{1, \cdots n\}$

as follows.

Given $y \in \{1, \cdots, n\}$.

$g(y) =$ the unique element x

in $\{1, \cdots, n\}$

s.t. $f(x) = y$.

Rk

x exists because f is surjective.

x is unique because f is injective.

Then check

g is also bijective.

&

$f \circ g = 1$

$g \circ f = 1$.

Example $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 7 & 3 & 8 & 1 & 5 & 2 & 6 \end{pmatrix}$

f^{-1}.

$f^{-1}(1)$ is the unique element

$x \in \{1, \dots, 8\}$

s.t. $f(x) = 1$.

What is this? The expression of f above tells you that's $\underline{5}$.

$f^{-1}(2)$ is $x \in \{1, \dots, 8\}$

s.t. $f(x) = 2$

That's $\underline{\underline{7}}$

$$
\begin{array}{ccc}
 & f^{-1} & \\
1 & \mapsto & 5 \\
2 & \mapsto & 7 \\
3 & \mapsto & 3 \\
4 & \mapsto & 1 \\
5 & \mapsto & 6 \\
6 & \mapsto & 8 \\
7 & \mapsto & 2 \\
8 & \mapsto & 4
\end{array}
$$

$$f^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 7 & 3 & 1 & 6 & 8 & 2 & 4 \end{pmatrix}$$