Recap quiz

Consider 2-player zero-sum game

$$\begin{array}{cc|ccc} & & & \text{Colin} & \\ & & c_1 & c_2 & c_3 \\ \hline \text{Rosemary} & r_1 & 1 & 2 & 3 \\ & r_2 & -3 & -2 & -1 \end{array}$$

Call this payoff matrix $A = a_{ij}$

What is the expected payoff to Rosemary/Colin if Rosemary plays mixed strategy $\underline{x} = (\frac{1}{2}, \frac{1}{2})$

Colin plays $\underline{y} = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$

Expected payoff to Rosemary

$$\underline{x}^T A \underline{y} = \frac{1}{2} \times \frac{1}{4} \times 1 + \frac{1}{2} \times \frac{1}{2} \times 2 + \frac{1}{2} \times \frac{1}{4} \times 3 + \frac{1}{2} \times \frac{1}{4} \times (-3) + \frac{1}{2} \times \frac{1}{2} \times (-2) + \frac{1}{2} \times \frac{1}{4} \times (-1) = 0$$

Expected payoff to Colin $= -0 = 0$

What is the expected payoff to Rosemary if

Rosemary plays mixed strategy $\underline{x} = (\frac{1}{4}, \frac{3}{4})$

Colin plays pure strategy c_3 i.e. $\underline{y} = (0, 0, 1)$

expected payoff to Rosemary $= \underline{x}^T A \underline{y}$

$$= \frac{1}{4} \times 1 \times 3 + \frac{3}{4} \times 1 \times -1 = (\underline{x}^T A)_3$$

Recall: Consider 2-player zero-sum game

with $R = \{r_1, \dots, r_k\}$ set of Rosemary's strategies

$C = \{c_1, \dots, c_\ell\}$ set of Colin's strategies

$A = a_{ij}$

If Rosemary plays mixed strategy $\underline{x} = (x_1, \dots, x_k) \in \Delta(R)$

Colin plays $\underline{y} = (y_1, \dots, y_\ell) \in \Delta(C)$

then expected payoff to Rosemary is $\underline{x}^T A \underline{y}$

Colin is $-\underline{x}^T A \underline{y}$

If Rosemary plays mixed strategy $\underline{x} \in \Delta(R)$

Colin plays pure strategy c_j

equivalently $\underline{y} = (0, \dots, 0, 1, 0, \dots, 0)$ $= \underline{e}_j$ jth entry

then expected payoff to Rosemary is $(\underline{x}^T A)\underline{e}_j = (\underline{x}^T A)_j$

Intuitively, the security level for $\underline{x} \in \Delta(R)$ is least expected payoff to Rosemary if she plays $\underline{x}$, i.e.

$$\min_{\underline{y} \in \Delta(C)} \underline{x}^T A \underline{y}$$

Difficult to use this to compute security of $\underline{x}$

Next lemma simplifies above expression,

Lem With previous setup

(1) For any fixed $\underline{x} \in \Delta(R)$

$$\min_{\underline{y} \in \Delta(C)} \underline{x}^T A \underline{y} = \min_{c_j \in C} (\underline{x}^T A)_j$$

smallest entry in $\underline{x}^T A$

(2) For any fixed $\underline{y} \in \Delta(C)$

$$\max_{\underline{x} \in \Delta(R)} \underline{x}^T A \underline{y} = \max_{r_i \in R} (A\underline{y})_i$$

largest entry in $A\underline{y}$

Pf (omitted)

Defn With previous setup

The <u>security level for Roseman's mixed strategy</u> $\underline{x} \in \Delta(R)$ is

$$\min_{c_j \in C} (\underline{x}^T A)_j = \min_{c_j \in C} \left(\sum_{i=1}^{k} x_i a_{ij} \right)$$

the <u>security level for Colin's mixed strategy</u> $\underline{y} \in \Delta(C)$ is

$$\max_{r_i \in R} (A\underline{y})_i = \max_{r_i \in R} \left(\sum_{j=1}^{\ell} a_{ij} y_j \right)$$

Defn With previous setup

the security level for Rosemary's mixed strategy $\underline{x} \in \Delta(R)$

$$\min_{c_j \in C} (\underline{x}^T A)_j = \min_{c_j \in C} \left(\sum_{i=1}^{k} x_i a_{ij} \right)$$

the security level for Colin's mixed strategy $\underline{y} \in \Delta(C)$

$$\max_{r_i \in R} (A\underline{y})_i = \max_{r_i \in R} \left(\sum_{j=1}^{l} a_{ij} y_j \right)$$

Example

Rosemary \ Colin	h	t
h	1	-1
t	-1	1

Find security level for Rosemary's mixed strategy.

$\underline{r} = (\frac{1}{3}, \frac{2}{3})$

Ans: expected payoff to Rosemary if Colin plays h is

$$\frac{1}{3} \times 1 + \frac{2}{3} \times (-1) = -\frac{1}{3}$$

expected payoff to Rosemary if Colin plays t is

$$\frac{1}{3} \times (-1) + \frac{2}{3} \times 1 = \frac{1}{3}$$

security level of $\underline{r}$ is $\min(-\frac{1}{3}, \frac{1}{3}) = -\frac{1}{3}$.

Recall: What is pure Nash equilibrium

$$\begin{array}{c|ccccccc} & c_1 & c_2 & \cdots & c_j & \cdots & c_\ell \\ \hline r_1 & & & & & & \\ r_2 & & & & & & \\ \vdots & & & & & & \\ r_i & & & & a_{ij} & & \\ \vdots & & & & & & \\ r_k & & & & & & \end{array}$$

In words

(r_i, c_j) is a pure Nash equilibrium if
$(\underline{x}, \underline{y})$ is a mixed Nash equilibrium if

Rosemary has no incentive to change her strategy $\overset{\underline{x}}{r_i}$
assuming Colin stays at $\overset{\underline{y}}{c_j}$ <u>and</u>

Colin has no incentive to change his strategy $\overset{\underline{y}}{c_j}$
assuming Rosemary stays at $\overset{\underline{x}}{r_i}$

expected payoff is $x^T A y$

<u>Defn</u> Consider a 2-player zero-sum game with payoff matrix A.
$(\underline{x}, \underline{y})$ with $\underline{x} \in \Delta(R)$ and $\underline{y} \in \Delta(C)$ is a <u>(mixed) Nash equilibrium</u> if

$$\underline{x}^T A \underline{y} \geq \underline{x}'^T A \underline{y} \quad \forall \underline{x}' \in \Delta(R)$$

and

$$\underline{x}^T A \underline{y} \leq \underline{x} A \underline{y}' \quad \forall \underline{y}' \in \Delta(C)$$

Following theorem makes it easier to check if $(\underline{x}, \underline{y})$ is a (mixed) Nash equilibrium.

<u>Thm</u> Given 2-player, zero-sum game with payoff matrix A, let $\underline{x} \in \Delta(R)$ and $\underline{y} \in \Delta(C)$ be mixed strategies, and let

$u(\underline{x})$ be security level for $\underline{x} = \min_j (\underline{x}^T A)_j$;
$u(\underline{y})$ be security level for $\underline{y} = \max_i (A\underline{y})_i$;

Then $(\underline{x}, \underline{y})$ is a (mixed) Nash equilibrium if and only if

$$u(\underline{x}) = u(\underline{y}).$$

<u>Example</u>

	c_1	c_2
r_1	1	3
r_2	4	2

Set $\underline{x} = (\frac{1}{2}, \frac{1}{2})$, $\underline{y} = (\frac{1}{4}, \frac{3}{4})$

Show $(\underline{x}, \underline{y})$ is Nash equilibrium

security for $\underline{x}$: payoff if Colin plays

c_1: $\frac{1}{2} \times 1 + \frac{1}{2} \times 4 = \frac{5}{2}$ $\qquad$ c_2: $\frac{1}{2} \times 3 + \frac{1}{2} \times 2 = \frac{5}{2}$

Security of $\underline{x} = \min(\frac{5}{2}, \frac{5}{2}) = \frac{5}{2}$.

security for $\underline{y}$: payoff if Roseman plays

r_1: $\frac{1}{4} \times 1 + \frac{3}{4} \times 3 = \frac{5}{2}$ $\qquad$ r_2: $\frac{1}{4} \times 4 + \frac{3}{4} \times 2 = \frac{5}{2}$

security of $\underline{y} = \max(\frac{5}{2}, \frac{5}{2}) = \frac{5}{2}$

$(\underline{x}, \underline{y})$ is a (mixed) Nash equilibrium by theorem above since security levels of $\underline{x}$ and $\underline{y}$ are equal.

Thm Given 2-player, zero-sum game with payoff matrix A, let $\underline{x} \in \Delta(R)$ and $\underline{y} \in \Delta(C)$ be mixed strategies, and let

$u(\underline{x})$ be security level for $\underline{x} = \min_j (\underline{x}^T A)_j$;

$u(\underline{y})$ be security level for $\underline{y} = \max_i (A\underline{y})_i$;

Then $(\underline{x}, \underline{y})$ is a Nash equilibrium if and only if

$$u(\underline{x}) = u(\underline{y}).$$

Pf
$$u(\underline{x}) \underset{\substack{\uparrow \\ \text{defn} \\ +\text{ lemma}}}{=} \min_{\underline{y}' \in \Delta(C)} \underline{x}^T A \underline{y}' \overset{①}{\leq} \underline{x}^T A \underline{y} \overset{②}{\leq} \max_{\underline{x}' \in \Delta(R)} \underline{x}'^T A \underline{y} \underset{\substack{\uparrow \\ \text{def} \\ +\text{ lemma}}}{=} u(\underline{y})$$

If $u(\underline{x}) = u(\underline{y})$ then ① and ② hold with equality

② with equality says Rosemary has no incentive to change from $\underline{x}$

① with equality says Colin has no incentive to change from $\underline{y}$

Hence $(\underline{x}, \underline{y})$ is a Nash equilibrium.

If $(\underline{x}, \underline{y})$ is (mixed) Nash equilibrium then

Rosemary has no incentive to change (ie ② holds with equality)

Colin has no incentive to change (i.e. ① holds with equality)

So $u(\underline{x}) = u(\underline{y})$.

$\square$

Know how to compute security level for a particular mixed strategy

How do we find mixed strategy with best security level?

optimal mixed strategy means mixed strategy with best security level

For Rosemary this is the mixed strategy with highest security and for Colin, the lowest.

Example 11.1. Give a linear program for finding the row player's optimal mixed strategy for the zero-sum game with the following payoff matrix:

Rosemary \ Colin	1	2
1	2	−3
2	−3	4
3	4	−5

Let $\underline{x} = (x_1, x_2, x_3)$ be a mixed strategy for Rosemary.

Security of $\underline{x}$: expected payoff if Colin plays 1
$= 2x_1 - 3x_2 + 3x_3$
expected payoff if Colin plays 2
$= -3x_1 + 4x_2 - 5x_3$

Security level of $\underline{x} = \min(2x_1 - 3x_2 + 3x_3, -3x_1 + 4x_2 - 5x_3)$

Want to find $\underline{x}$ with maximum security level

$$\begin{array}{ll} \text{maximise} & \min(2x_1 - 3x_2 + 3x_3,\ -3x_1 + 4x_2 - 5x_3) \\ \text{subject to} & x_1 + x_2 + x_3 = 1 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Equivalent LP (see end of week 8)

$$\begin{array}{ll} \text{maximise} & z \\ \text{sub to} & z \leq 2x_1 - 3x_2 + 3x_3 \\ & z \leq -3x_1 + 4x_2 - 5x_3 \\ & x_1 + x_2 + x_3 = 1 \\ & x_1, x_2, x_3 \geq 0, \ z \text{ unrestricted.} \end{array}$$

Optimal solution (x_1, x_2, x_3, z) to this LP gives the mixed strategy (x_1, x_2, x_3) with highest security level for Rosemary, and z gives security level of that mixed strategy

Write LP to find Colin's mixed strategy with best (lowest) security level.

Example 11.1. Give a linear program for finding the row player's optimal mixed strategy for the zero-sum game with the following payoff matrix:

		Colin	
		1	2
Rosemary	1	2	−3
	2	−3	4
	3	4	−5

Colin : let (y_1, y_2) be mixed strategy for Colin.

expected payoff if Rosemary plays 1 : $2y_1 - 3y_2$

2 : $-3y_1 + 4y_2$

3 : $4y_1 - 5y_2$

security level for $\underline{y}$ = $\max(2y_1 - 3y_2, -3y_1 + 4y_2, 4y_1 - 5y_2)$

$\underline{y}$ with best (lowest) security given by

$$\min_{\underline{y} \in \Delta(C)} \max(2y_1 - 3y_2, -3y_1 + 4y_2, 4y_1 - 5y_2)$$

LP : minimise t

sub to

$t \geq 2y_1 - 3y_2$

$t \geq -3y_1 + 4y_2$

$t \geq 4y_1 - 5y_2$

$y_1 + y_2 + y_3 = 1$

$y_1, y_2, y_3 \geq 0$, t unrestricted.

Rosemary's and Colin's LPs turn out to be dual of each other.

In general if Rosemary has strategies $R = \{r_1, \dots, r_n\}$

Colin $C = \{c_1, \dots, c_m\}$

$A = a_{ij}$ is payoff matrix

If Rosemary plays $\underline{x} = (x_1, \dots, x_n) \in \Delta(R)$

and Colin plays c_j

Expected payoff to Rosemary $= a_{1j} x_1 + a_{2j} x_2 + \cdots + a_{nj} x_n$

Security of $\underline{x}$ is $u(\underline{x}) = \min\limits_{c_j \in C} (a_{1j} x_1 + a_{2j} x_2 + \cdots + a_{nj} x_n)$

Best security level for Rosemary

$$\max_{\underline{x} \in \Delta(R)} u(\underline{x}) = \max_{\underline{x} \in \Delta(R)} \left(\min_{c_j \in C} a_{1j} x_1 + a_{2j} x_2 + \cdots + a_{nj} x_n \right)$$

i.e. solve $\max \left[\min\limits_{c_j \in C} a_{1j} x_1 + a_{2j} x_2 + \cdots + a_{nj} x_j \right]$

Sub to $x_1 + x_2 + \cdots + x_n = 1$

$x_1, x_2, \dots x_n \geq 0$

Equivalent LP is

maximize z

sub to $z \leq a_{1j} x_1 + a_{2j} x_2 + \cdots + a_{nj} x_j \quad j = 1, \dots, m$

$x_1 + x_2 + \cdots + x_n = 1$

$x_1, x_2, \dots, x_n \geq 0$, z unrestricted.

$n+1$ variables

$m+1$ constraints.

Can similarly check that best security level for Colin given by following LP

$$\begin{aligned} &\text{minimise } t \\ &\text{sub to} \quad t \geq a_{i1}y_1 + a_{i2}y_2 + \dots + a_{im}y_m \quad i = 1, \dots, n \\ &\qquad\quad y_1 + y_2 + \dots + y_m = 1 \\ &\qquad\quad y_1, y_2, \dots, y_m \geq 0, \quad t \text{ unrestricted.} \end{aligned}$$

$m+1$ variables
$n+1$ constraints

Proposition Rosemary's and Colin's LPs for finding best security levels are dual to each other.

Suppose $(x_1, \dots, x_n, z)$ is optimal solution to Rosemary's LP

$(y_1, \dots, y_m, t)$ is optimal solution to Colin's LP

then $\underset{\underline{x}}{(x_1, \dots, x_n)}$ is mixed strategy for Rosemary with best security level $z = u(\underline{x})$

and $\underset{\underline{y}}{(y_1, \dots, y_m)}$ is mixed strategy for Colin with best security level $t = u(\underline{y})$

Strong duality theorem says $z = t$ i.e. $u(\underline{x}) = u(\underline{y})$
$\Rightarrow$ (by earlier thm) $(\underline{x}, \underline{y})$ is a Nash equilibrium

Have proved

<u>Thm</u> Every 2-player zero-sum game has a (mixed) Nash equilibrium.

<u>Remarks</u>

- For 2-player zero-sum games, have seen that we sometimes have no pure Nash equilibrium, but always have a mixed Nash equilibrium. i.e. can always get stability if we allow random strategies.
- In fact, John Nash showed that any 2-player game (not necessarily zero-sum) has a mixed Nash equilibrium. We will not prove this, but will look again briefly at general 2-player games next week.