MTH5114 Linear Programming and Games, Spring 2024

Week 10 Seminar Questions Viresh Patel

Practice Exam Question:

Consider the following 2-player zero-sum game. Each player separately chooses a number from the set $\{1, 2, 3\}$. Both players then reveal their numbers. If the numbers match, the row player must pay £3 to the column player, otherwise, the player with the lower number must pay £1 to the player with the higher number.

(a) Give the payoff matrix for this game from the perspective of the row player. Also give the security level for each of the player's strategies.

Solution:

	1	2	3
1	-3	-1	-1
2	1	-3	-1
3	1	1	-3

The security levels for the row player's strategies (1, 2, and 3) are -3, -3 and -3, respectively. The security levels for the column player's strategies are 1, 1, and -1, respectively.

(b) Does this game possess a pure Nash equilibrium? If so, give all pure Nash equilibria for the game. If not, say why.

Solution: The game does not possess a pure Nash equilibrium, because the maximum security level for the row player is -3 but the minimum security level for the column player is -1. At a Nash equilibrium, these two quantities must be equal.

(c) Formulate a linear program that finds the row player's best mixed strategy in this game (you do not need to solve this program). [You will be able to do this part of the question at the end of week 11.]

Discussion Questions:

1. (a) Give an example of a 2-player zero-sum game in which the row player (Rosemary) has 2 strategies, the column player (Colin) has 3 strategies and every pair of strategies is a Nash equilibrium (i.e. there are six Nash equilibria.) Exhibit your example by giving the payoff matrix (as usual from the row player's perspective).

Solution: The only example where this can happen is if all six entries of the payoff matrix have the same value.

(b) Following on from the previous question, for each $n = 0, 1, \ldots, 6$ give an example of a 2-player zero-sum game with the following properties or explain why one does not exist. The row player (Rosemary) has 2 strategies, the column player (Colin) has 3 strategies and there are exactly n Nash equilibria. Note that the previous question is the case $n = 6$ of this question.

Solution:

- (n=0)

	c_1	c_2	c_3
r_1	1	-1	0
r_2	-1	1	0

- (n=1)

	c_1	c_2	c_3
r_1	0	1	1
r_2	-1	0	0

- (n=2)

	c_1	c_2	c_3
r_1	0	1	1
r_2	0	1	1

- (n=3)

	c_1	c_2	c_3
r_1	1	1	1
r_2	0	0	0

- (n=4)

	c_1	c_2	c_3
r_1	0	0	1
r_2	0	0	1

- (n=5) There is no example here. Suppose there was an example with exactly 5 Nash equilibria. Any two Nash equilibria in the same row or the same column must have the same payoff (using the definition of Nash equilibrium). Using this we can deduce that all 5 Nash equilibria have the same payoff, say x. The remaining payoff in the payoff matrix (call it y) cannot be equal to x or else we would have six Nash equilibria. If $y > x$ then the row player prefers to move from x to y and if $y < x$ then the column player prefers to move from x to y; in each of these cases, we see that not all outcomes with payoff x are Nash equilibria, so in fact we have fewer than 5. [Other correct explanations are possible.]
- (n=6) Done in previous question.

2. Consider an arbitrary 2-player zero-sum game where Rosemary's set of strategies is $\{r_1, r_2\}$ and Colin's set of strategies is $\{c_1, c_2\}$, and the payoff to Rosemary when Rosemary plays r_i and Colin plays c_j is the number $a_{ij} \in \mathbb{R}$.

(a) Write down the payoff matrix for this game.

Solution:

	c_1	c_2
r_1	a_{11}	a_{12}
r_2	a_{21}	a_{22}

(b) Suppose that Rosemary uses a mixed strategy $(p, 1-p)$ and Colin uses a mixed strategy $(q, 1-q)$, where $p, q \in [0, 1]$. What is the expected payoff to Rosemary and what is the payoff to Colin.

Solution: The expected payoff to Rosemary is given by

$$\begin{pmatrix} p & 1-p \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} q \\ 1-q \end{pmatrix} = a_{11}pq + a_{12}p(1-q) + a_{21}(1-p)q + a_{22}(1-p)(1-q).$$

The expected payoff to Colin is the negative of this.

(c) By writing Colin's mixed strategy as

$$\begin{pmatrix} q \\ 1-q \end{pmatrix} = q \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (1-q) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

show that Colin can improve[1] his expected payoff by using one of his pure strategies instead of his mixed strategy $\mathbf{q} = (q, (1-q))$ assuming that Rosemary sticks with her mixed strategy $\mathbf{p} = (p, (1-p))$.

Solution: Let's write A for the payoff matrix. Let's write $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

In the previous part, we saw that if Rosemary plays the mixed strategy $\mathbf{p}$ and Colin plays the mixed strategy $\mathbf{q}$ then the expected payoff to Rosemary is $\mathbf{p}^\intercal A\mathbf{q}$ and the expected payoff to Colin is $-\mathbf{p}^\intercal A\mathbf{q}$. We know we can write $\mathbf{q} = q\mathbf{e}_1 + (1-q)\mathbf{e}_2$, so the expected payoff to Colin (when Rosemary plays $\mathbf{p}$ and Colin plays $\mathbf{q}$) is

$$-\mathbf{p}^\intercal A\mathbf{q} = -\mathbf{p}^\intercal A(q\mathbf{e}_1 + (1-q)\mathbf{e}_2) = -q\mathbf{p}^\intercal A\mathbf{e}_1 - (1-q)\mathbf{p}^\intercal A\mathbf{e}_2$$

Notice that in the last expression $-\mathbf{p}^\intercal A\mathbf{e}_1$ is the expected payoff to Colin when Rosemary plays $\mathbf{p}$ and Colin plays the pure strategy c_1 and similarly $-\mathbf{p}^\intercal A\mathbf{e}_2$ is the expected payoff to Colin when Rosemary plays $\mathbf{p}$ and Colin plays the pure strategy c_2. Choose the higher of these two expected payoffs and call Colin's pure strategy with this higher expected payoff c_i. In other words

$$-\mathbf{p}^\intercal A\mathbf{e}_i = \max(-\mathbf{p}^\intercal A\mathbf{e}_1, -\mathbf{p}^\intercal A\mathbf{e}_2).$$

Then, continuing the expression from above

$$\begin{aligned} -\mathbf{p}^\intercal A\mathbf{q} = -\mathbf{p}^\intercal A(q\mathbf{e}_1 + (1-q)\mathbf{e}_2) &= -q\mathbf{p}^\intercal A\mathbf{e}_1 - (1-q)\mathbf{p}^\intercal A\mathbf{e}_2 \\ &\leq -q\mathbf{p}^\intercal A\mathbf{e}_i - (1-q)\mathbf{p}^\intercal A\mathbf{e}_i = \mathbf{p}^\intercal A\mathbf{e}_i. \end{aligned}$$

This shows that Colin's payoff is at least as good when he plays the pure strategy c_i instead of the mixed strategy $\mathbf{q}$ (assuming Rosemary plays $\mathbf{p}$ is both cases).

[1] Here, by improve, we mean "at least as good as" so might not be a strict improvement