Example 10.1.8 Consider the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n,$$

let $y = x-1$, then the radius of convergence of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} y^n$ is $R=1$ (by the Ratio Test).

Hence the interval of convergence for x is $(0,2)$ together (possibly with $x=0$ and/or $x=2$).

For $x=0$, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(0-1)^n = -\sum_{n=1}^{\infty} \frac{1}{n}$ which is divergent. $(=-\infty)$

For $x=2$, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(2-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln(1+x)\Big|_{x=1} = \ln(2)$

$\approx 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

$\therefore$ Interval of convergence is $(0,2]$.

Alt series $\sum (-1)^n a_n$ $a_n \to 0$ and alternating, then $\sum (-1)^n a_n$ C

Exercises

RT = Ratio Test

CRT = Cauchy Root Test

(i) $f_n(x) = \dfrac{n}{n+1} x^n$ $\quad R = 1, \; x_0 = 0 \quad$ RT

(ii) $f_n(x) = \dfrac{4^{1+2n}}{5^{n+1}} x^n$ $\quad R = \dfrac{5}{16}, \; x_0 = 0 \quad$ CRT

(iii) $f_n(x) = \dfrac{4^{1+2n}}{5^{n+1}} (x+3)^n$ $\quad R = \dfrac{5}{16}, \; x_0 = -3 \quad$ CRT

(iv) $f_n(x) = \dfrac{(n+1)}{(2n+1)!} x^n$ $\quad R = \infty, \; x_0 = 0 \quad$ RT

(v) $f_n(x) = \dfrac{n+1}{(2n+1)!} (x-2)^n$ $\quad R = \infty, \; x_0 = 2 \quad$ RT

§10.2 Differentiating & Integrating Power Series

Thm 10.2.1 (Continuity)

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R, then f converges uniformly on $[-R_1, R_1]$ & the power series is continuous for $R_1 \in (0, R)$.

Proof By Theorem 10.1.2, the ratio Test gives

$$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{|a_{n+1}|}{|a_n|}\right|,$$

so $\sum a_n x^n$ and $\sum |a_n| x^n$ have the same radius of convergence.

Since $0 < R_1 < R$, we have $\sum_{n=0}^{\infty} a_n x^n$, $|x| \le R_1$ converges uniformly by the M-test, using $\sum_{n=0}^{\infty} |a_n| R_1^n$ converges, and

$$|a_n x^n| \le \left||a_n| x^n\right| \le \left||a_n| R_1^n\right|$$

$\sum_{n=0}^{\infty} |a_n x^n|$ C. by comparison with $\sum_{n=0}^{\infty} |a_n| R_1^n$

Corollary 10.2.2 The power series $\sum_{n=0}^{\infty} a_n x^n$ converges to a continuous function on the open interval $(-R,R)$.

Proof If $x_0 \in (-R,R)$, $\exists$ some $R_1 < R$, $x \in (-R_1, R_1) \subset (-R,R)$

and then by Thm 10.2.1, the limit of the series is continuous (uniform limit of a sequence of continuous functions is continuous).

Thm 9.1.6

Comment We emphasise that the series sum may not converge uniformly on the full radius of convergence $(-R,R)$.

Lemma 10.2.3

If the power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R, then $\sum_{n=1}^{\infty} n a_n x^{n-1}$ (D) and $\sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$ (I) also have the same radius of convergence (RoC)

Proof First observe the "differential" series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ has RoC $R_d = R$

$$\beta_d = \frac{1}{R_d} = \lim_{n\to\infty} |n a_n|^{1/n} = \lim_{n\to\infty} n^{1/n} |a_n|^{1/n} = \lim_{n\to\infty} |a_n|^{1/n} = \frac{1}{R} = \beta$$

The "integral" series $\sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1} = x \sum_{n=0}^{\infty} \frac{a_n x^n}{n+1}$ has RoC $R_I = R$

$$\beta_I = \frac{1}{R_I} = \lim_{n\to\infty} \left|\frac{a_n}{n+1}\right|^{1/n} = \lim_{n\to\infty} \frac{|a_n|^{1/n}}{(n+1)^{1/n}} = \lim_{n\to\infty} |a_n|^{1/n} = \frac{1}{R} = \beta$$

$\therefore$ $R_I = R$ and $R_d = R$.

$$\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} \frac{1}{(n+1)^{1/n}} = 1$$

LECTURE NOTES COMPLETED