Tutorial Seminars

MTH5105

Weeks 11 – 12

1. Consider the following sequence of functions f_n.

   (a) $f_n(x) = \frac{x}{n}$,
   (b) $f_n(x) = \frac{1}{1+x^n}$,
   (c) $f_n(x) = \frac{x^n}{1+x^n}$.

   For each of the functions above consider the following

   (i) Find $f(x) = \lim_{n\to\infty} f_n(x)$,
   (ii) Determine whether $f_n \to f$ uniformly on $[0, 1]$,
   (ii) Determine whether $f_n \to f$ uniformly on $\mathbb{R}$.

(a) $f_n(x) = \frac{x}{n}$; $f_n(x) \to 0$ as $n \to \infty$ $\forall x \in \mathbb{R}$

(i) $f(x) \equiv 0$

(ii) For $x \in [0,1]$: $|f_n(x) - f(x)| = \frac{|x|}{n} \leq \frac{1}{n}$,

Given $\varepsilon > 0$:

$\frac{1}{n} < \varepsilon$, $n > N(\varepsilon) = \frac{1}{\varepsilon}$ and so

$|f_n(x) - f(x)| < \varepsilon$

for $n > N(\varepsilon)$ $\therefore$ U.C.

(iii) For $x \in \mathbb{R}$: Not u.c.

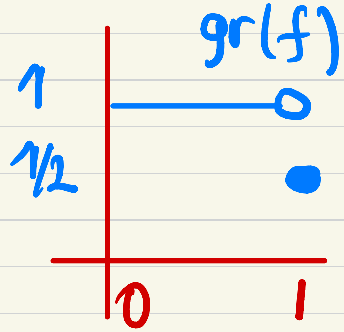
suppose $\varepsilon = \frac{1}{2}$, for any n, $f_n(n) = \frac{n}{n} = 1$

so $\nexists$ N s.t. $|f_n(x) - f(x)| < \varepsilon$ $\forall n > N$.

because $f_n(n) = 1$.

(b) $f_n(x) = \dfrac{1}{1+x^n}$

(i) $\lim_{n\to\infty} |x|^n = 0, \ |x|<1; \ 1, \ |x|=1; \ \infty, \ |x|>1$



$x \in [0,1)$ $\quad$ $x=1$

$\therefore \ f_n(x) \to 1, \ |x|<1; \ \frac{1}{2}, \ x=1; \ 0, \ |x|>1; \ x=-1$ (NC)

(ii) UC on $[0,1]$: No $f(x)$ discontinuous and f_n's cont.

Thm: UConv continuous f_n's give a continuous limit function

(iii) UC on $\mathbb{R}$: No! because not uniformly conv. on $[0,1] \subseteq \mathbb{R}$.

If NOT UConv on a subset $\Omega \subseteq \mathbb{R}$, then NOT UConv on any superset of Ω.

(c) $f_n(x) = \dfrac{x^n}{1+x^n}$ As in (b) re. x^n

(i) $f_n(x) \to 0, |x|<1$; $1, |x|>1$; $\frac{1}{2}, x=1$. Not defined $x=-1$.

(ii) UC on $[0,1]$: No f discontinuous ; f_n's continuous

x^n cont. and $1+x^n$ cont., $\dfrac{x^n}{1+x^n}$ cont.

(iii) Two independent reasons

UC on $\mathbb{R}$ No, because f_n not UC on $[0,1]$.

Limit function not defined at $x=-1$, $\forall n$.

$x \neq 0$ $\quad \dfrac{x^n}{1+x^n} = \dfrac{1}{1+\frac{1}{x^n}} \longrightarrow 1$ as $n \to \infty$.

2. Let $f_n(x) = x, g_n(x) = \frac{1}{n}$ for all $x \in \mathbb{R}$. Let $f(x) = x, g(x) = 0$.

(a) Show that $f_n \to x, g_n \to 0$ uniformly,

(b) Does $f_n g_n \to fg$ uniformly?

$f_n(x) = x \qquad \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x = x \; (= f(x))$

UC? Consider $|f_n(x) - f(x)| = x - x = 0$!

Given $\varepsilon > 0$ $|f_n(x) - f(x)| < \varepsilon$ for all n ! $\therefore$ UC ✓

$g_n(x) = \frac{1}{n} \qquad \lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \frac{1}{n} = 0. \; (= g(x))$

UC? Consider $|g_n(x) - g(x)| = \left|\frac{1}{n} - 0\right|$

Given $\varepsilon > 0$, $|g_n(x) - g(x)| = \frac{1}{n} < \varepsilon$ for $n > \frac{1}{\varepsilon}$.

$\therefore g_n$ UC.

Now consider $f_n \cdot g_n$ i.e. $(f_n \cdot g_n)(x) = x \cdot \frac{1}{n} = \frac{x}{n}$.

$$\lim_{n\to\infty} (f_n \circ g_n)(x) = 0 \quad \text{for} \quad x \in \mathbb{R}.$$

UC. Given $\varepsilon > 0$, does $\exists N$ s.t. for $n > N$

$$|(f_n \circ g_n)(x) - \underbrace{f\circ g(x)}_{=0}| < \varepsilon, \quad \forall x \in \mathbb{R}$$

Counterexample let $\varepsilon = \frac{1}{2}$; consider

$$|(f_n \circ g_n)(n) - f(n)| = \left|\frac{n}{n}\right| = 1 \not< \frac{1}{2} \quad \forall n.$$

3. Let $\{f_n\}$ be a sequence of integrable functions on $[a, b]$ and suppose that $f_n \to f$ uniformly on $[a, b]$. Prove that f is integrable and that

$$\int_a^b f\,dx = \lim_{n\to\infty} \int_a^b f_n.$$

Check Notes!

$f_n \xrightarrow{UC} f$ on $[a,b]$. Given $\varepsilon > 0$, $\exists N$ s.t.

for $n > N$ $\quad |f_n(x) - f(x)| < \cancel{\varepsilon}\ \frac{\varepsilon}{b-a}*$ $\forall x \in [a,b]$.

$$-|f_n(x) - f(x)| < f_n(x) - f(x) < |f_n(x) - f(x)|$$

$$\Rightarrow -\int_a^b |f_n(x) - f(x)|\,dx < \int_a^b f_n(x) - f(x)\,dx \le \int_a^b |f_n(x) - f(x)|\,dx$$

$$\therefore \left| \int_a^b (f_n(x) - f(x))\,dx \right| < \frac{\varepsilon}{(b-a)}^{*}(b-a) = \varepsilon$$

$$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| < \varepsilon, \quad n > N \qquad \text{i.e.} \quad \lim_{n\to\infty} \int_a^b f_n(x) = \int_a^b f(x)\,dx.$$

4. Let $\{g_n\}$ be the sequence of functions $g_n : \mathbb{R} \to \mathbb{R}, n \in \mathbb{N}$ given by

$$g_n(x) = \frac{x}{1 + nx^2}.$$

(a) Compute $g(x) = \lim_{n\to\infty} g_n(x)$.

(b) Show g_n converges to g uniformly.

(c) Compute $h(x) = \lim_{n\to\infty} g_n'(x)$.

(d) Does $g'(x) = h(x)$ hold?

$$g_n(x) = \frac{x}{1+nx^2}$$

$$g_n(0) = 0$$

For $x \neq 0$ $\quad |g_n(x)| = \left|\frac{x}{1+nx^2}\right| \leq \frac{|x|}{|nx^2|}$

As $n \to \infty$ $\quad \frac{|x|}{|nx^2|} \to 0$ as $n \to \infty$

$$\therefore g_n(x) \to 0 \text{ as } n \to \infty$$

$\therefore g(x) = 0$

$$|g_n(x) - g(x)| = |g_n(x)| = \frac{|x|}{1+n|x|^2} < \frac{1}{n|x|}, \quad \forall x \in \mathbb{R}.$$

Given $\varepsilon > 0$, $\quad |g_n(x) - g(x)| < \varepsilon$ provided

$\frac{1}{n|x|} < \varepsilon$ i.e. $n > \frac{1}{\varepsilon |x|} = N$? Not good ! $\quad N$ depends on $|x|$

So we try an alternative - the Weierstrass M-test.

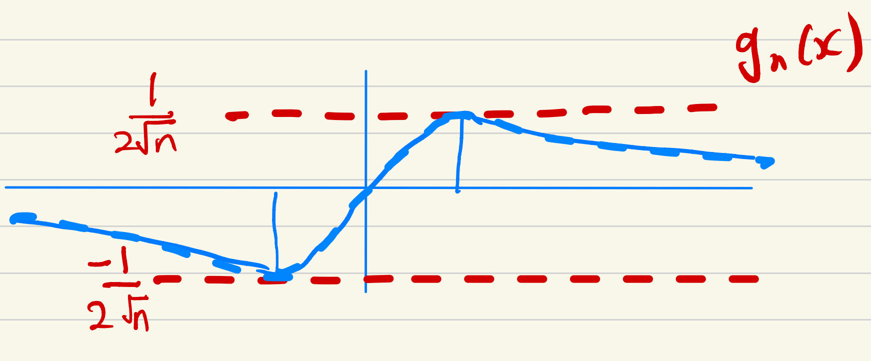
$$\text{Max} \frac{|x|}{1+n|x|^2} \leq$$

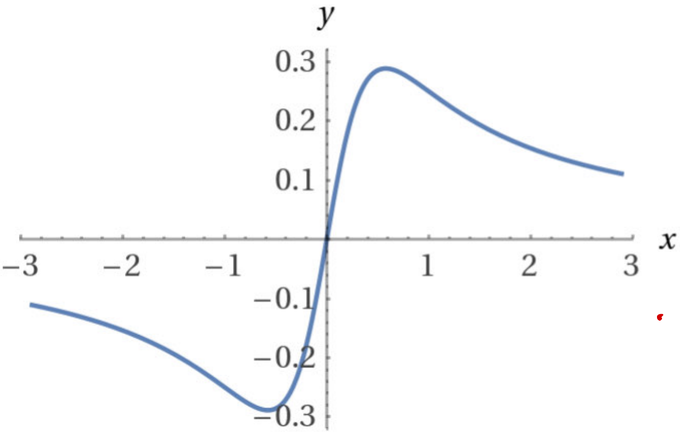
$$\frac{d}{dx}\left(\frac{x}{1+nx^2}\right) = \frac{(1+nx^2).1 - 2nx.x}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2} \quad (*)$$

$$= 0 \quad \text{at} \quad x^2 = 1/n \quad , \; x = \pm \frac{1}{\sqrt{n}}$$

Max/Min of $g_n(x)$ is:

$$\pm \frac{\frac{1}{\sqrt{n}}}{1+n\left(\frac{1}{\sqrt{n}}\right)^2} = \pm \frac{\frac{1}{\sqrt{n}}}{2} = \pm \frac{1}{2\sqrt{n}}$$





(x from −2.9 to 2.9)

$\therefore\ |g_n(x)| \le \frac{1}{2\sqrt{n}}$, $\forall x \in \mathbb{R}$ and so $g(x) \equiv 0$ since $\frac{1}{2\sqrt{n}} \to 0$ as $n \to \infty$

Note $|g_n(x) - 0| < \frac{1}{2\sqrt{n}} < \varepsilon$ for $n > \frac{1}{(2\varepsilon)^2}$ $\left(= N(\varepsilon)\right)$.

$$g_n'(x) = \frac{1 - nx^2}{(1 + nx^2)^2}$$ (see earlier *)

$$|g_n'(x)| \le \frac{nx^2}{(nx^2)^2} = \frac{1}{n}\frac{1}{x^2} \to 0 \text{ as } n \to \infty \quad \text{for } x \neq 0.$$

$$g_n'(0) \equiv 1 \qquad \therefore\ h(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$$

Conclusion $h(x) \neq g'(x)$.

5. Show that the Taylor series of a polynomial

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

is precisely that polynomial.

$$f(x) = a_0 + a_1 x \cdots + a_n x^n = \sum_{k=0}^{n} a_k x^k$$

$$f(0) = a_0$$

$$f^{(1)}(x) = \sum_{k=1}^{n} k a_k x^{k-1} \Rightarrow f^{(1)}(0) = 1 a_1$$

$$f^{(2)}(x) = \sum_{k=2}^{n} k(k-1) x^{k-2} \Rightarrow f^{(2)}(0) = 2 \cdot 1 \cdot a_2$$

$$\vdots$$

$$f^{(i)}(x) = \sum_{k=i}^{n} k(k-1) \cdots (k-(i-1)) x^{k-i} \Rightarrow f^{(i)}(0) = i!\, a_i$$

$$\vdots$$

$$f^{(n)}(x) = \sum_{k=n}^{n} n(n-1) \cdots 2 \cdot 1\, x^{k-n} \Rightarrow f^{(n)}(0) = n!\, a_n$$

$$\therefore T_{(n,0)} f(x) = \sum_{k=0}^{n} \frac{f^{k}(0)\, x^k}{k!} = \sum_{k=0}^{n} \frac{k!\, a_k}{k!} x^k = \sum_{k=0}^{n} a_k x^k$$

$$R_{(n,0)} f(x) = f^{(n+1)}(c) \frac{x^{n+1}}{(n+1)!}, \quad \text{Note } f^{(n+1)}(x) \equiv 0, \ \forall x \in \mathbb{R}$$

$$\therefore R_{(n,0)} f(x) \equiv 0$$

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