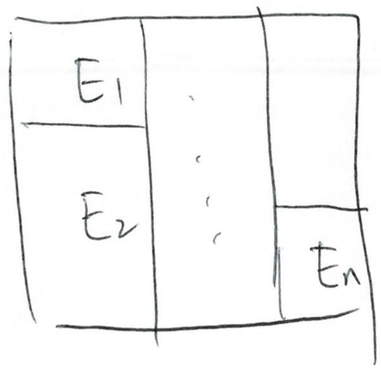

17.5 Conditional expectation

Proposition 17.3 (Properties of conditional expectation)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

ξ, ζ : r.v.s

$\mathcal{G}, \mathcal{G}$ be two sub-σ-algebras of $\mathcal{F}$.

Then

Exp. I For any real numbers a, b

$$E[a\xi + b\zeta \mid \mathcal{G}] = aE[\xi \mid \mathcal{G}] + bE[\zeta \mid \mathcal{G}].$$ Linearity

Exp. 2 $E[E[\xi | G]] = E[\xi]$

Exp. 3 If ζ is a r.v. measurable w.r.t G,

then $E[\xi \zeta | G] = \zeta E[\xi | G]$. $\boxed{\text{Past}}$

Ecomonic meaning: Stock price of yesterday

is known today, no random

info up to today

Exp. 4 If ζ is a r.v. independent of G $\boxed{\text{future}}$

then $E[\zeta | G] = E[\zeta]$ math: $P(A|B) = P(A)$ if $A \perp B$

Eco: Stock price of tomorrow

Complete market

no arbitrage

Exp 5. if $\mathcal{G}_1 \subset \mathcal{G}$

then $E[\xi \mid \mathcal{G}_1] = E\left[E[\xi \mid \mathcal{G}] \mid \mathcal{G}_1 \right]$

Public info + insider info   Public info

Example:

Let ξ_i, $i = 1, 2, \dots, n$ independent r.v.s ± 1 $P(\xi_i = 1) = \frac{1}{2}$

$\mathcal{F}_n$ the σ-algebra generated by $\xi_1, \xi_2, \dots, \xi_n$

$S_n = \sum_{i=1}^{n} \xi_i$

τ: r.v taking values in $\mathbb{N}$, with $E(\tau) < \infty$

$\tau \perp$ all ξ_i

1. $E\left[e^{\xi_1 + \xi_2 - \xi_3} \mid \xi_1, \xi_2 \right]$

$\overset{\text{Exp.3}}{=} e^{\xi_1 + \xi_2} E\left[e^{-\xi_3} \mid \xi_1, \xi_2 \right]$

$\overset{\text{Exp.4}}{=} e^{\xi_1 + \xi_2} E\left[e^{-\xi_3} \right]$ all ξ_i s are independent

$E\left[e^{\mp \xi_3} \right] = e^{-1} \times \frac{1}{2} + e^{+1} \times \frac{1}{2} = \cosh(1)$

$= e^{\xi_1 + \xi_2} \cosh(1)$

2. $E\left[S_n \mid \mathcal{F}_{n-1} \right]$ → all the info up to time $n-1$

$= E\left[S_{n-1} + \xi_n \mid \mathcal{F}_{n-1} \right]$

$\overset{\text{Exp1}}{=} E\left[S_{n-1} \mid \mathcal{F}_{n-1} \right] + E\left[\xi_n \mid \mathcal{F}_{n-1} \right]$

past

$\overset{\text{Exp3}}{=} S_{n-1} + E\left[\xi_n \mid \mathcal{F}_{n-1} \right] \overset{\text{Exp4}}{=} S_{n-1} + E\left[\xi_n \right]$

$E(\xi_n) = 1 \times \frac{1}{2} + (-1) \times \frac{1}{2} = 0$

$\underbrace{E[S_n \mid \mathcal{F}_{n-1}]}_{\text{info time } n-1} = S_{n-1} + 0 = \underbrace{S_{n-1}}$

best estimate: Stock price of current time

$E(\xi_n) = 0$

3. $E[S_n^2 - n \mid \mathcal{F}_{n-1}]$

~~Hytho~~ Hypothesis: $E[S_n^2 - n \mid \mathcal{F}_{n-1}] = S_{n-1}^2 - (n-1)$

Martingale

$\overset{\text{Exp 1}}{=} E[S_n^2 \mid \mathcal{F}_{n-1}] - n$

$= E[(S_{n-1} + \xi_n)^2 \mid \mathcal{F}_{n-1}] - n$

$\overset{\text{Exp 1}}{=} E[S_{n-1}^2 \mid \mathcal{F}_{n-1}] + 2E[S_{n-1}\xi_n \mid \mathcal{F}_{n-1}] + E[\xi_n^2 \mid \mathcal{F}_{n-1}] - n$

$\overset{\text{Exp 3}}{=} S_{n-1}^2 + 2S_{n-1}E[\xi_n \mid \mathcal{F}_{n-1}] + E[\xi_n^2 \mid \mathcal{F}_{n-1}] - n$

$\overset{\text{Exp 4}}{=} S_{n-1}^2 + 2S_{n-1}E[\xi_n] + E[\xi_n^2] - n$

$E[\xi_n^2] = 1 \times \frac{1}{2} + (-1)^2 \times \frac{1}{2} = 1$

$= S_{n-1}^2 - (n-1)$

4. $E[e^{S_n} \mid \tilde{\mathcal{F}}_{n-1}]$ $\qquad S_n = S_{n-1} + \xi_n$

$= E[e^{S_{n-1}} e^{\xi_n} \mid \tilde{\mathcal{F}}_{n-1}]$

$\overset{\text{Exp3}}{=} e^{S_{n-1}} E[e^{\xi_n} \mid \tilde{\mathcal{F}}_{n-1}]$

$\overset{\text{Exp4}}{=} e^{S_{n-1}} E[e^{\xi_n}] = e^{S_{n-1}} \cosh(1)$

5. $E[S_n^2 \mid S_{n-1}]$

$= E[(S_{n-1} + \xi_n)^2 \mid S_{n-1}] \overset{\text{Exp1}}{=} E[S_{n-1}^2 \mid S_{n-1}] + 2E[S_{n-1}\xi_n \mid S_{n-1}] + E[\xi_n^2 \mid S_{n-1}]$

$\overset{\text{Exp3}}{=} S_{n-1}^2 + 2S_{n-1} E[\xi_n \mid S_{n-1}] + E[\xi_n^2 \mid S_{n-1}]$

$\overset{\text{Exp4}}{=} S_{n-1}^2 + 2S_{n-1} E[\xi_n] + E[\xi_n^2]$

$= S_{n-1}^2 + 1$

$\neq S_{n-1}^2$

6. $E[S_\tau^2 \mid \tau]$ τ : r.v. in $\mathbb{N}$, $E[\tau] < \infty$

$$E[S_n^2] = E[(\xi_1 + \xi_2 + \cdots + \xi_n)^2] = E\Big[\sum_{j,k} \xi_j \xi_k\Big]$$

ξ_i s are independent

$$= \sum_j E[\xi_j^2] + \sum_{j \neq k} E[\xi_j \xi_k] \text{ ~~independent~~ } = \sum_j E[\xi_j^2] + \sum_{j \neq k} E[\xi_j] E[\xi_k]$$

$E(AB) = E(A) \cdot E(B)$

Independence

$$= \sum_j E[\xi_j^2] = n$$

$$E[S_\tau^2 \mid \tau = n] = n = \tau$$

$$E[S_\tau^2 \mid \tau] = \tau$$

$$E[\xi_i] = 1 \times \frac{1}{2} + (-1) \times \frac{1}{2} = 0$$

$$E[\xi_i^2] = 1^2 \times \frac{1}{2} + (-1)^2 \times \frac{1}{2} = 1$$

W11 ⑦

18. Martingale

Def Filtration $\tilde{\mathcal{F}}_t$ adapted Slide 41

Defination 18.3 Martingale

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $\{\tilde{\mathcal{F}}_t\}_{t\geq 0}$

and an adapted process $\{X_t\}_{t\geq 0}$

$\hookrightarrow$ $\tilde{\mathcal{F}}_t$ - measurable

The process is a martingale if

- M.1. for any $t \geq 0$ $E[|X_t|] < \infty$
- M.2. for any $\underset{\text{future}}{t} \geq \underset{\text{current}}{s}$ $\boxed{E[X_t | \tilde{\mathcal{F}}_s] = X_s.}$

In discrete setting

$\{\tilde{\mathcal{F}}_n\}_{n\in\mathbb{N}}$ $\tilde{\mathcal{F}}_n \subseteq \tilde{\mathcal{F}}_{n+1}$

If an adapted process is a martingale,

M.2 $E[X_n \mid \mathcal{F}_{n-1}] = X_{n-1}$

Exp. 5 if $\mathcal{G}_1 \subset \mathcal{G}$, then $E[\xi \mid \mathcal{G}_1] = E[E[\xi \mid \mathcal{G}] \mid \mathcal{G}_1]$

Tower rule:

$$E[X_n \mid \mathcal{F}_m] = E[E[E[\underbrace{E[X_n \mid \mathcal{F}_{n-1}] \mid \mathcal{F}_{n-2}}] \mid \cdots] \mid \mathcal{F}_m]$$

$$\overset{\text{Exp5}}{=} E[E[E[X_{n-1} \mid \mathcal{F}_{n-2}] \mid \cdots] \mid \mathcal{F}_m]$$

$$= E[E[X_{n-1} \mid \cdots] \mid \mathcal{F}_m] = \cdots = X_m$$

Example 1.

$$E(W_t \mid \tilde{\mathcal{F}}_s) = E[W_t - W_s + W_s \mid \tilde{\mathcal{F}}_s]$$

$$\overset{\text{Exp 1}}{=} E[W_t - W_s \mid \tilde{\mathcal{F}}_s] + E[W_s \mid \tilde{\mathcal{F}}_s]$$

$$\overset{\text{Exp 3}}{=} E[W_t - W_s \mid \tilde{\mathcal{F}}_s] + W_s$$

$$\overset{\text{Exp 4}}{=} E[W_t - W_s] + W_s = W_s$$

M.2. $E[X_t \mid \tilde{\mathcal{F}}_s] = X_s$

$$W_t - W_s \sim N(0, t-s)$$

$$B_t = B_0 + \mu t + \sigma W_t$$

$$\tilde{E}[B_t \mid \tilde{\mathcal{F}}_s] = \tilde{E}[B_t - B_s + B_s \mid \tilde{\mathcal{F}}_s]$$

$$\overset{\text{Exp 1}}{=} \tilde{E}[B_t - B_s \mid \tilde{\mathcal{F}}_s] + \tilde{E}[B_s \mid \tilde{\mathcal{F}}_s]$$

$$\overset{\text{Exp 3}}{=} \tilde{E}[B_t - B_s \mid \tilde{\mathcal{F}}_s] + B_s$$

$$\overset{\text{Exp 4}}{=} \tilde{E}[B_t - B_s] + B_s$$

$$= \underset{\neq 0}{\mu}\,(\underbrace{t-s}_{\neq 0}) + B_s \neq B_s \quad \text{for } \mu \neq 0$$

Example 2

$$E[W_t^2 | \tilde{\mathcal{F}}_s] = E[(W_t - W_s + W_s)^2 | \tilde{\mathcal{F}}_s]$$

$$\overset{\text{Exp 1.}}{=} E[(W_t - W_s)^2 | \tilde{\mathcal{F}}_s] + 2E[W_s (W_t - W_s) | \tilde{\mathcal{F}}_s] + E[W_s^2 | \tilde{\mathcal{F}}_s]$$

$$\overset{\text{Exp. 3}}{=} E[(W_t - W_s)^2 | \tilde{\mathcal{F}}_s] + 2W_s E[W_t - W_s | \tilde{\mathcal{F}}_s] + W_s^2$$

$$\overset{\text{Exp. 4}}{=} E[(W_t - W_s)^2] + 2W_s E[W_t - W_s] + W_s^2$$

$$E(W_t^2 | \tilde{\mathcal{F}}_s) = t - s + W_s^2$$

W_t^2 is not a martingale

$$E[W_t^2 - t | \tilde{\mathcal{F}}_s] = W_s^2 - s$$

$W_t^2 - t$ is a martingale w.r.t $\{\tilde{\mathcal{F}}_t\}_{t \geq 0}$

$W_t - W_s \sim N(0, t-s)$

$Var[(W_t - W_s)]$

$= t - s$

$E[W_t - W_s] = 0$

$E[(W_t - W_s)^2]$

$= Var[W_t - W_s] + (E[W_t - W_s])^2$

Example 3. $S_t = e^{B_t}$. S_t is GBM $S_0 = e^{B_0}$

$$B_t = B_0 + \mu t + \sigma W_t$$

Q: Under what conditions can S_t be a martingale?

Hint: $\frac{S_t}{S_s} \cdot S_s$

A. $E[S_t \mid \tilde{\mathcal{F}}_s] = E\left[\boxed{\frac{S_t}{S_s}} \cdot S_s \mid \tilde{\mathcal{F}}_s\right] = E\left[e^{\mu(t-s)+\sigma(W_t - W_s)} \cdot S_s \mid \tilde{\mathcal{F}}_s\right]$

$$= e^{\mu(t-s)} E\left[e^{\sigma(W_t - W_s)} \cdot S_s \mid \tilde{\mathcal{F}}_s\right]$$

$$\overset{\text{Exp 3}}{=} e^{\mu(t-s)} \cdot S_s \, E\left[e^{\sigma(W_t - W_s)} \mid \tilde{\mathcal{F}}_s\right]$$

$$\overset{\text{Exp 4}}{=} e^{\mu(t-s)} S_s \, \underbrace{E\left[e^{\sigma(W_t - W_s)}\right]}$$

$$= e^{\mu(t-s)} S_s \cdot e^{\frac{\sigma^2}{2}(t-s)} = S_s \, e^{\mu(t-s) + \frac{\sigma^2}{2}(t-s)}$$

$$E[S_t \mid \tilde{\mathcal{F}}_s] = S_s \, e^{(\mu + \frac{\sigma^2}{2})(t-s)}$$

$S_t = e^{B_t} = e^{B_0 + \mu t + \sigma W_t}$

$\frac{S_t}{S_s} = e^{\mu(t-s) + \sigma(W_t - W_s)}$

$$E[S_t \mid \tilde{F}_s] \overset{||}{=} S_s$$

$$S_s \, e^{(\mu+\frac{\sigma^2}{2})(\underset{\neq 0}{t-s})} = S_s$$

$= 1$

$$\mu + \frac{\sigma^2}{2} = 0 \qquad \mu = -\frac{\sigma^2}{2}$$

Prove: $X_t = e^{-rt} S_t$ is a martingale

$S_t \sim$ risk-neutral GBM

Hint: $S_t = S e^{(r-\frac{1}{2}\sigma^2)t + \sigma W_t}$

An important property of Martingale: constant mean

Proposition 18.1 (Constant mean of martingale)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space

X_t : martingale w.r.t $\{\mathcal{F}_t\}_{t \geq 0}$

Then the mean $E[X_t]$ is constant over time,

$$E[X_t] = E[X_s], \quad \forall\, t, s \geq 0$$

Proof: $t > s$

Exp 2. $E[E[\xi|\mathcal{G}]] = E[\xi]$

$$E[X_t] \overset{\text{Exp. 2}}{=} E[\underbrace{E[X_t | \mathcal{F}_s]}_{= X_s \text{ martingale}}] = E[X_s] \quad \boxtimes$$

$e^{-rt} S_t$ constant mean

$$E[e^{-rt} S_t] = E[e^{-r \times 0} \cdot S_0] = S_0$$

$$E[S_t] = e^{rt} S_0$$

r - RNP