Queen Mary
University of London

Main Examination period 2019

MTH4115 / MTH4215: Vectors & Matrices

Duration: 2 hours

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Examiners: O. Jenkinson, R. Johnson

Question 1. [20 marks] Let A, B, C be points in 3-space with respective position vectors $\mathbf{a} = \begin{pmatrix}1\\0\\2\end{pmatrix}$, $\mathbf{b} = \begin{pmatrix}2\\3\\4\end{pmatrix}$, $\mathbf{c} = \begin{pmatrix}-1\\-3\\-2\end{pmatrix}$. Determine:

(a) The length of the vector $3\mathbf{a} - \mathbf{b}$; [**3**]

(b) A unit vector in the direction of $\mathbf{b}$; [**3**]

(c) $\mathbf{a} \cdot \mathbf{b}$; [**3**]

(d) $\mathbf{a} \times \mathbf{b}$; [**3**]

(e) A vector equation for the line through A and B; [**4**]

(f) The coordinates of the point D such that $ABCD$ is a parallelogram. [**4**]

Solutions 1.

(a) $3\mathbf{a} - \mathbf{b} = \begin{pmatrix}1\\-3\\2\end{pmatrix}$, which has length $\sqrt{1+9+4} = \sqrt{14}$.

(b) $\mathbf{b}$ has length $\sqrt{2^2+3^2+4^2} = \sqrt{29}$, so a unit vector in the direction of $\mathbf{b}$ is $\frac{1}{\sqrt{29}}\mathbf{b} = \begin{pmatrix}2/\sqrt{29}\\3/\sqrt{29}\\4/\sqrt{29}\end{pmatrix}$

(c) $\mathbf{a} \cdot \mathbf{b} = 1 \times 2 + 0 \times 3 + 2 \times 4 = 10$

(d) $\mathbf{a} \times \mathbf{b} = \begin{pmatrix}-6\\0\\3\end{pmatrix}$

(e) $\mathbf{b} - \mathbf{a} = \begin{pmatrix}1\\3\\2\end{pmatrix}$, so a vector equation is $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) = \begin{pmatrix}1\\0\\2\end{pmatrix} + \lambda \begin{pmatrix}1\\3\\2\end{pmatrix}$.

(f) Let $\mathbf{d}$ be the position vector for D. For $ABCD$ to be a parallelogram we need $\mathbf{c} - \mathbf{d} = \mathbf{b} - \mathbf{a}$, so $\mathbf{d} = \mathbf{a} - \mathbf{b} + \mathbf{c} = \begin{pmatrix}1\\0\\2\end{pmatrix} - \begin{pmatrix}2\\3\\4\end{pmatrix} + \begin{pmatrix}-1\\-3\\-2\end{pmatrix} = \begin{pmatrix}-2\\-6\\-4\end{pmatrix}$, so D has coordinates $(-2, -6, -4)$.

Question 2. [20 marks] Suppose that vectors $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ are given.

(a) Write down an expression for the **scalar product** $\mathbf{u} \cdot \mathbf{v}$ (in terms of the coordinates of $\mathbf{u}$ and $\mathbf{v}$). **[3]**

(b) What does it mean to say that two vectors are **orthogonal**? **[3]**

(c) Show that if a vector is orthogonal to all vectors, then it must be the zero vector. **[4]**

(d) How is the **vector product** $\mathbf{u} \times \mathbf{v}$ defined (in terms of the coordinates of $\mathbf{u}$ and $\mathbf{v}$)? **[3]**

(e) Show that $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$. **[3]**

(f) Show that if $\mathbf{u}$ has the property that $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ for all vectors $\mathbf{v}$, then necessarily $\mathbf{u} = \mathbf{0}$. **[4]**

Solutions 2.

(a) $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$

(b) Two vectors $\mathbf{u}$, $\mathbf{v}$ are **orthogonal** if $\mathbf{u} \cdot \mathbf{v} = 0$.

(c) If $\mathbf{u}$ is such that $\mathbf{u} \cdot \mathbf{v} = 0$ for all $\mathbf{v}$, then in particular $|\mathbf{u}|^2 = \mathbf{u} \cdot \mathbf{u} = 0$, so $|\mathbf{u}| = 0$, and the only vector with zero length is the zero vector, so $\mathbf{u} = \mathbf{0}$.

(d) $\mathbf{u} \times \mathbf{v} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}$.

(e)

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = (u_2 v_3 - u_3 v_2) u_1 + (u_3 v_1 - u_1 v_3) u_2 + (u_1 v_2 - u_2 v_1) u_3$$

and we see that the three terms $u_1 u_2 v_3$, $u_1 u_3 v_2$, $u_2 u_3 v_1$ each occur twice in this expression, once with coefficient $+1$ and once with coefficient -1. The terms therefore cancel in pairs, so the expression reduces to 0, whence the orthogonality.

(f) We might use the result (proved in lectures) that $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$ where θ is the angle between (non-zero) vectors $\mathbf{u}$, $\mathbf{v}$, so if $\mathbf{v}$ is chosen to be non-zero and with $\sin\theta > 0$ (i.e. v is not a scalar multiple of $\mathbf{u}$) then $|\mathbf{u}| = |\mathbf{u} \times \mathbf{v}| / (|\mathbf{v}| \sin\theta) = 0$, so $\mathbf{u}$ has zero length and therefore must be the zero vector.

Alternatively, from the formula in (d) we see that $\mathbf{0} = \mathbf{u} \times \mathbf{i} = \begin{pmatrix} 0 \\ u_3 \\ -u_2 \end{pmatrix}$, so $u_2 = u_3 = 0$, and $\mathbf{0} = \mathbf{u} \times \mathbf{j} = \begin{pmatrix} -u_3 \\ 0 \\ u_1 \end{pmatrix}$, so $u_1 = 0$; therefore $u_1 = u_2 = u_3 = 0$, and hence $\mathbf{u} = \mathbf{0}$.

Question 3. [20 marks] Let Π_1 be the x-y plane (i.e. with equation $z = 0$), let Π_2 be the x-z plane (i.e. with equation $y = 0$), let Π_3 be the y-z plane (i.e. with equation $x = 0$), and let Π_4 be the plane with equation $x + y + z = 1$. Let Q be the point with position vector $\mathbf{q} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$.

(a) Determine the distance between Q and Π_1. **[2]**

(b) Determine the distance between Q and Π_4. **[3]**

(c) Determine the coordinates of the point on Π_4 that is closest to Q. **[3]**

(d) If A denotes the point in the intersection $\Pi_1 \cap \Pi_2 \cap \Pi_4$, and B denotes the point in the intersection $\Pi_1 \cap \Pi_3 \cap \Pi_4$, determine the coordinates of the mid-point C of A and B. **[3]**

(e) If l denotes the line through the points C (from part (d) above) and Q, then determine the coordinates of the point in the intersection $l \cap \Pi_3$. **[4]**

(f) Determine the coordinates of a point which is equidistant from the four planes Π_1, Π_2, Π_3, Π_4 (i.e. the point has the same distance from each of these planes). **[5]**

Solutions 3.

(a) The distance is 1 (i.e. the z-component of $\mathbf{q}$)

(b) The vector $\mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is orthogonal to Π_4, so this distance (using the formula derived in lectures) is $|\mathbf{q}\cdot\mathbf{n} - 1|/|\mathbf{n}| = 1/\sqrt{3}$.

(c) Using the formula from lectures, this closest point has position vector

$$\mathbf{q} - \left(\frac{\mathbf{q}\cdot\mathbf{n}-1}{|\mathbf{n}|^2}\right)\mathbf{n} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} - (-1/3)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -8/3 \\ 7/3 \\ 4/3 \end{pmatrix},$$

so its coordinates are $(-8/3, 7/3, 4/3)$.

(d) The point A has coordinates $(1,0,0)$, and the point B has coordinates $(0,1,0)$, so the mid-point C has coordinates $(1/2, 1/2, 0)$.

(e) The direction of l is given by $\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} -7/2 \\ 3/2 \\ 1 \end{pmatrix}$, so an equation for l is $\mathbf{r} = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -7/2 \\ 3/2 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$. The line l intersects Π_3 when $x = 0$, i.e. when $\frac{1}{2} - \frac{7}{2}\lambda = 0$, i.e. $\lambda = 1/7$. So the point of intersection has coordinates $(0, \frac{5}{7}, \frac{1}{7})$.

(f) To be equidistant from Π_1, Π_2 and Π_3 means the position vector of the point must be of the form $a\mathbf{n} = \begin{pmatrix} a \\ a \\ a \end{pmatrix}$ for some $a \in \mathbb{R}$, and the common distance to these planes is a. We need that its distance to Π_4 also equals a, in other words that

$$a = \frac{|a\mathbf{n}\cdot\mathbf{n} - 1|}{|\mathbf{n}|} = \frac{|3a-1|}{\sqrt{3}},$$

and this equation has the two solutions $a = 1/(3-\sqrt{3})$ and $a = 1/(3+\sqrt{3})$.

So one solution is the point with coordinates $(\frac{1}{3-\sqrt{3}}, \frac{1}{3-\sqrt{3}}, \frac{1}{3-\sqrt{3}})$, another solution is the point with coordinates $(\frac{1}{3+\sqrt{3}}, \frac{1}{3+\sqrt{3}}, \frac{1}{3+\sqrt{3}})$.

Question 4. [20 marks] Consider the linear system

$$\begin{array}{rcrcrcrcl} x_1 & - & 2x_2 & + & x_3 & - & x_4 & = & 0 \\ 2x_1 & - & 3x_2 & + & 4x_3 & - & 3x_4 & = & 0 \\ -x_1 & + & x_2 & - & 3x_3 & + & 2x_4 & = & 0 \end{array} .$$

(a) Write down the augmented matrix of the system. **[3]**

(b) Bring the augmented matrix to reduced row echelon form, indicating the elementary row operations used at each step. **[4]**

(c) Identify the leading and the free variables, and write down the solution set of the system. **[4]**

(d) Let l_1, l_2 and l_3 be lines in 3-space, such that l_1 passes through $(1,4,-3)$ in the direction $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, l_2 passes through $(1,3,-2)$ in the direction $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$, and l_3 passes through $(2,6,-4)$ in the direction $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$.

Write down parametric equations for each of these three lines. **[3]**

(e) For the lines l_1, l_2, l_3 as in part (d) above, determine the intersection $l_1 \cap l_2$ of l_1 and l_2, the intersection $l_1 \cap l_3$ of l_1 and l_3, and the intersection $l_2 \cap l_3$ of l_2 and l_3. **[6]**

Solutions 4.

(a)

$$\left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 2 & -3 & 4 & -3 & 0 \\ -1 & 1 & -3 & 2 & 0 \end{array}\right).$$

(b) Using elementary row operations we find:

$$\left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 2 & -3 & 4 & -3 & 0 \\ -1 & 1 & -3 & 2 & 0 \end{array}\right) \sim \begin{array}{c} R_2 - 2R_1 \\ R_3 + R_1 \end{array} \left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & -1 & -2 & 1 & 0 \end{array}\right)$$

$$\sim_{R_3 + R_2} \left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \sim_{R_1 + 2R_2} \left(\begin{array}{cccc|c} 1 & 0 & 5 & -3 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

where the last matrix is now in reduced row echelon form.

(c) Leading variables: x_1, x_2. Free variables: x_3, x_4.

Let $x_4 = \alpha$ and $x_3 = \beta$. Then $x_1 = 3\alpha - 5\beta$ and $x_2 = \alpha - 2\beta$. So the solution set can be written as

$$\{(3\alpha - 5\beta, \alpha - 2\beta, \beta, \alpha) : \alpha, \beta \in \mathbb{R}\}.$$

(d) The line l_1 has parametric equations

$$\left.\begin{array}{r} x = 1 + \lambda \\ y = 4 + 2\lambda \\ z = -3 - \lambda \end{array}\right\},$$

the line l_2 has parametric equations

$$\left.\begin{array}{r} x = 1 + 2\mu \\ y = 3 + 3\mu \\ z = -2 - \mu \end{array}\right\},$$

and the line l_3 has parametric equations

$$\left.\begin{array}{r} x = 2 + 2\nu \\ y = 6 + 3\nu \\ z = -4 - \nu \end{array}\right\}.$$

(e) The lines l_2 and l_3 are parallel and distinct, so $l_2 \cap l_3$ is the empty set.

The lines l_1 and l_2 do intersect, with $l_1 \cap l_2 = \{(-1, 0, -1)\}$. This could be computed directly by equating the parametric equations for l_1 and l_2 to find $\lambda = -2$, $\mu = -1$ (or alternatively we could find λ, μ by setting $(x_1, x_2, x_3, x_4) = (\lambda, \mu, 1, 1)$ in the system from parts (a)–(c)).

The lines l_1 and l_3 do intersect, with $l_1 \cap l_3 = \{(2, 6, -4)\}$. This could be computed directly by equating the parametric equations for l_1 and l_3 to find $\lambda = 1$, $\nu = 0$ (or alternatively we could find λ, ν by setting $(x_1, x_2, x_3, x_4) = (\lambda, \nu, 1, 2)$ in the system from parts (a)–(c)).

Question 5. [20 marks] Let

$$A = \begin{pmatrix} 1 & 3 \\ -2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 0 \\ 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 0 & 0 & 3 \\ 9 & 0 & 1 & 8 \\ -8 & 2 & 4 & 5 \\ 3 & 0 & 0 & 5 \end{pmatrix}.$$

(a) For each of the products A^2, AB, BA, B^2, BC, CB, state whether or not it exists; if it exists then evaluate it. **[6]**

(b) Explain what it means for a matrix M to be **invertible**, and what is meant by the **inverse** of M. **[4]**

(c) Calculate $\det(C)$ and decide whether C is invertible or not. **[4]**

(d) Using part (c) above, evaluate $\det(C^6)$ and $\det(3C)$. In each case, briefly explain which property of determinants you are using. **[4]**

(e) Find $\det(D)$, where D is the matrix obtained from C by subtracting 13 times column 1 from column 4. Briefly explain which property of determinants you are using. **[2]**

Solutions 5.

(a) The products AB, B^2, and BC do not exist, but the other three do exist, with

$$A^2 = \begin{pmatrix} -5 & 3 \\ -2 & -6 \end{pmatrix}, \quad BA = \begin{pmatrix} -2 & 0 \\ 1 & 3 \\ 2 & 6 \\ 2 & 0 \end{pmatrix}, \quad CB = \begin{pmatrix} 0 & -1 \\ 2 & 1 \\ 10 & -13 \\ 0 & -2 \end{pmatrix}.$$

(b) A (necessarily square) matrix M is called **invertible** if it has an inverse. To say that N is the **inverse** of N means that $MN = NM = I$ (the identity matrix).

(c)

$$\det(C) = \begin{vmatrix} 2 & 0 & 0 & 3 \\ 9 & 0 & 1 & 8 \\ -8 & 2 & 4 & 5 \\ 3 & 0 & 0 & 5 \end{vmatrix} = -2 \begin{vmatrix} 2 & 0 & 3 \\ 9 & 1 & 8 \\ 3 & 0 & 5 \end{vmatrix} = (-2)\cdot 1 \cdot \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = (-2)\cdot 1 \cdot (10-9) = -2.$$

Since $\det(C) = -2 \neq 0$, the matrix C is invertible.

(d) The determinant is multiplicative (i.e. $\det(MN) = \det(M)\det(N)$), so $\det(C^6) = (\det(C))^6 = (-2)^6 = 64$.

Since $3C$ is obtained from C by multiplying each of the 4 rows by 3, it follows that

$$\det(3C) = 3^4 \det(C) = -2 \cdot 3^4 = -162.$$

(e) Since determinants are not changed by elementary column operations of type III we have $\det(D) = \det(C) = -2$.

End of Paper.