CHAPTER 9

Connected Spaces

9.1. Connected and disconnected topological spaces

DEFINITION 9.1. A topological space X is said to be *connected* if any continuous map $f : X \to \{0, 1\}$ is constant. Here $\{0, 1\} \subset \mathbf{R}$ is equipped with the induced (discrete) topology.

A space which is not connected is called *disconnected*.

If a topological space X is homeomorphic to a connected topological space Y then X is connected as well. This directly follows from the definition.

LEMMA 9.2. *The following conditions are equivalent:*

(a) *X is connected;*
(b) *There are no continuous surjective maps $f : X \to \{0, 1\}$;*
(c) *Any subset $U \subset X$ which is both closed and open is either $U = \emptyset$ or $U = X$.*

PROOF. Conditions (a) and (b) are clearly equivalent. If $f : X \to \{0, 1\}$ is a continuous surjective map then $f^{-1}(0) = U \subset X$ is a closed and open subset distinct from $\emptyset$, X, i.e. (b) implies (c). Conversely, if $U \subset X$ is a closed and open subset distinct from $\emptyset$, X then the map $f : X \to \{0, 1\}$ with $f(U) \equiv 0$ and $f(X - U) \equiv 1$ is continuous and surjective. □

LEMMA 9.3. *The intervals (a, b), $(a, b]$, $[a, b)$, $[a, b]$ are connected.*

PROOF. Any continuous map $f : (a, b) \to \{0, 1\}$ is constant as follows from the Mean Value Theorem from the course of Calculus. Similarly for all other intervals. □

LEMMA 9.4. *If $X = A \cup B$ where $A \cap B \neq \emptyset$ and A and B are connected then X is connected.*

PROOF. Let $f : X \to \{0, 1\}$ be continuous. Then the restrictions $f|_A$ and $f|_B$ are constant, and, since $A \cap B \neq \emptyset$, we see that f is constant. □

THEOREM 9.5. *Let $A \subset X$ be a connected subspace. If $A \subset B \subset \overline{A}$ then B is also connected.*

PROOF. Let $f : B \to \{0, 1\}$ be a continuous map. Then $f|_A$ is constant. Let's assume that $f|_A \equiv 0$. Then $U = f^{-1}(0) \subset B$ is a closed subset containing A and hence $U = B$, i.e. $f \equiv 0$. □

As a special case of Theorem 9.5 we obtain:

COROLLARY 9.6. If a subset A of a topological space X is connected then its closure $\overline{A} \subset X$ is connected as well.

EXAMPLE 9.7. The following spaces are disconnected:

(1) $\{0, 1\}$ is disconnected;
(2) $\mathbf{R} - \{0\}$ is disconnected;
(3) $\mathbb{Q}$ is disconnected;

(4) The Cantor set C is disconnected.

Next we define the notion of a *connected component*.

Let X be a topological space. We may introduce an equivalence relation on X by declaring points $x, y \in X$ to be equivalent $x \sim y$ if there exists a connected subset $A \subset X$ containing both x and y. This relation is clearly symmetric and reflexive. To show that it is transitive, assume that $x \sim y$ and $y \sim z$. Then $x, y \in A$ and $y, z \in B$ where $A, B \subset X$ are connected subsets. Then $C = A \cup B$ is connected (by Lemma 9.4) and $x, z \in C$, i.e. $x \sim z$.

The equivalence relation above defines a partition of the space X into equivalence classes which are called the connected components. Each of the connected components is a maximal (with respect to the inclusion $\subset$) connected subspace of X. Different connected components are disjoint.

From Corollary 9.6 we see that *every connected component is closed.*

Example 9.8. (1) Clearly, the only connected component of a connected topological space X is the whole X.

(2) The space $\mathbf{R} - \{0\}$ has two connected components $(-\infty, 0)$ and $(0, \infty)$.

(3) The Jordan Curve Theorem [**1**] states that the complement of a planar simple closed curve $C \subset \mathbf{R}^2$ (i.e. the set $\mathbf{R}^2 - C$) has two connected components (the interior and the exterior). Here $C = f(S^1)$ is the image of an injective continuous map $f : S^1 \to \mathbf{R}^2$ where S^1 is the standard circle.

9.2. Connected subspaces of R

Theorem 9.9. *A nonempty subset $A \subset \mathbf{R}$ is connected if and only if it is one of the intervals $(a, b), (a, b], [a, b), [a, b]$ where $a \in \mathbf{R} \cup \{-\infty\}$ and $b \in \mathbf{R} \cup \{\infty\}$.*

Proof. From Lemma 9.3 we know that the intervals are connected; we only need to prove the inverse statement, i.e. any connected subset of $\mathbf{R}$ is an interval.

Let $A \subset \mathbf{R}$ be connected and $a, b \in A$ where $a < b$. If $c \in (a, b)$ and $c \notin A$ we may consider the subsets $A \cap (-\infty, c) = U$ and $A \cap (c, \infty) = V$ are open and closed in A, they form a partition

$$A = U \sqcup V,$$

and $U \neq \emptyset$, $V \neq \emptyset$. This contradicts the connectivity of A, see Lemma 9.2.

The above argument shows that a connected subset $A \subset \mathbf{R}$ has the following property: if $a, b \in A$, where $a < b$, then $[a, b] \subset A$.

Denote $\alpha = \inf A$ and $\beta = \sup A$. If $c \in (\alpha, \beta)$ then (using properties of the infimum and supremum) one can find $a_n \in A \cap (\alpha, c)$ and $b_n \in A \cap (c, \beta)$. Applying the property of A described in the previous paragraph one obtains that $c \in A$. In other words, we obtain that

$$(\alpha, \beta) \subset A.$$

On the other hand obviously $A \subset [\alpha, \beta]$. Thus we see that A equals one of the intervals (α, β), $(\alpha, \beta]$, $[\alpha, \beta)$, $[\alpha, \beta]$. □

Theorem 9.10. *The image of a connected space under a continuous map is connected.*

Proof. Assume that X is connected and $f : X \to Y$ be a continuous and surjective map. If $\phi : Y \to \{0, 1\}$ is a continuous and surjective map that $\phi \circ f : X \to \{0, 1\}$ is also continuous and surjective which contradicts the connectivity of X. □

Corollary 9.11 (Intermediate Value Theorem). Let $f : X \to \mathbf{R}$ be a continuous function on a connected space X. Then the image $f(X) \subset \mathbf{R}$ is an interval.

Proof. By Theorem 9.10 the image $f(X)$ is connected and by Theorem 9.9 the image is an interval. □

We note also the following:

Example 9.12. Any non-empty connected subset of the Cantor set C is a single point. Indeed, if $A \subset C$ is a connected subset containing at least two points $a < b$ the by Theorem 9.9 C must contain the interval $[a, b]$. But this would contradict Theorem 4.33 sating that the Lebesgue measure of C is zero. We see that the Cantor set is *totally disconnected*, which means that its every connected component is a single point.

9.3. Path-connectedness

In this section we shall discuss a more intuitive notion of connectivity.

Definition 9.13. A topological space X is said to be *path-connected* if for any pair of points $x, y \in X$ there exists a continuous path $\gamma : [0, 1] \to X$ satisfying $\gamma(0) = x$ and $\gamma(1) = y$, see Figure 1.

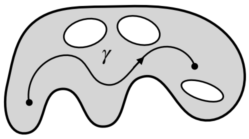

Figure 1. A path connecting two points in X.

Theorem 9.14. *Any path-connected space is connected.*

Proof. Assuming that X is path-connected, consider a continuous map $f : X \to \{0, 1\}$. If f is surjective, let $f(x) = 0$ and $f(y) = 1$, where $x, y \in X$. Applying Definition 9.13 we may find a continuous map $\gamma : [0, 1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. Then $f \circ \gamma : [0, 1] \to \{0, 1\}$ is a surjective continuous map; this contradicts Lemma 9.3 stating that the interval $[0, 1]$ is connected. □

Example 9.15. The following spaces are connected and path-connected:

(1) $\mathbf{R}^n$;
(2) Any convex subset of $\mathbf{R}^n$;
(3) A Banach space or a Hilbert space;
(4) The space ℓ^2;
(5) $\mathbf{R}^n - \{0\}$ if $n \geq 2$;
(6) $\mathbf{R}^n - A$, where A is a finite subset, if $n \geq 2$;
(7) $\mathbf{R}^n - L$, where $L \subset \mathbf{R}^n$ is a linear subspace, assuming that $n - \dim L \geq 2$.

Example 9.16. The following space $X \subset \mathbf{R}^2$ is connected but not path-connected. The space X is the union

$$X = A \cup B$$

where $A = 0 \times [-1, 1]$ and

$$B = \{(x, \sin(\frac{1}{x})); x \in (0, \infty)\},$$

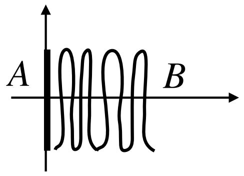

FIGURE 2. A connected but not path-connected space.

see Figure 2. Clearly the sets A and B are connected since they are homeomorphic to intervals, see Lemma 9.3. To show that X is connected we assume that $f : X \to \{0, 1\}$ be a continuous and surjective. The restrictions $f|_A$ and $f|_B$ are constant and their values are distinct since otherwise f would not be surjective. Assuming that $f|_A \equiv 0$ and $f|_B \equiv 1$ we see that the sequence

$$x_n = ((\pi n)^{-1}, 0) \in B$$

where $n = 1, 2, \dots$ satisfies $f(x_n) = 1$ while $f(0, 0) = 0$ and $(0, 0) = \lim x_n$, i.e. the function f is discontinuous.

To show that X is not path-connected, assume that $\gamma : [0, 1] \to X$ is a continuous path satisfying $\gamma(0) = (0, 0) \in A \subset X$ and $\gamma(1) = (\pi^{-1}, 0) = x_1 \in B \subset X$. Clearly, removing any point x_n makes B and X disconnected. Therefore, we can find a sequence

$$1 = t_1 > t_2 > t_3 > \cdots > 0$$

such that $\gamma(t_n) = x_n$ for $n = 1, 2, \dots$. The decreasing and bounded sequence t_n has a limit $t^* = \lim t_n$. If $p_1 : X \to \mathbf{R}$ and $p_2 : X \to \mathbf{R}$ denote the projections of X onto the x_1 and x_2 axes correspondingly, then for any n one has $p_1(\gamma(t_n)) = (\pi n)^{-1}$ and $p_2(\gamma(t_n)) = 0$ implying that $\gamma(t^*) = (0, 0)$. Let $U \subset \mathbf{R}^2$ be an open ball around $(0, 0)$ of radius $r < 1$. By continuity there exists $\delta > 0$ such that $\gamma(t) \in U$ for all $t \in (t^* - \delta, t^* + \delta) \cap [0, 1]$. But the set $U \cap X$ (see Figure 3) is obviously disconnected and the points x_n and x_{n+1} lie in different connected components. This

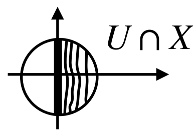

FIGURE 3. The set $U \cap X$.

contradiction shows that X is not path-connected.

9.4. Local connectedness and local path-connectedness

DEFINITION 9.17. A topological space X is said to be *locally connected at a point* $x \in X$ if every neighbourhood U of x contains a connected neighbourhood V of x.

A topological space X is *locally connected* if it is locally connected at each of its points.

EXAMPLE 9.18. The real line $\mathbf{R}$ is locally connected.

An open subset of a locally connected space is also locally connected.

The space $X = \{n^{-1}; n = 1, 2, \dots\} \cup \{0\} \subset \mathbf{R}$ is not locally connected.

THEOREM 9.19. *Every connected component of a locally connected space is open.*

PROOF. Let $C \subset X$ be a connected component of a locally connected space X. For $x \in C$, consider a connected open subset $U_x \subset X$ containing x. This set U_x must lie in C, i.e. $U_x \subset C$. Thus we see that C is open as it equals the union of the open sets U_x, where $x \in C$. □

Similarly to Definition 9.17 we introduce:

DEFINITION 9.20. A topological space X is said to be *locally path-connected at a point* $x \in X$ if every neighbourhood U of x contains a path-connected neighbourhood V of x.

A topological space X is *locally path-connected* if it is locally path-connected at each of its points.

EXAMPLE 9.21. The space X of Example 9.16 is not locally path-connected at the points $(0, y)$ where $y \in [-1, 1]$. Figure 3 shows the intersection of X with a small open ball; this intersection contains infinitely many path-connected components "converging" to $U \cap (0 \times [-1, 1])$. The same argument shows that X is not locally connected.

THEOREM 9.22. *Every path-connected component of a locally path-connected space is open.*

The proof is similar to the proof of Theorem 9.19.

THEOREM 9.23. *A locally path-connected space is path-connected if and only if it is connencted.*

PROOF. We only need to show that for locally path-connected spaces connectedness implies path-connectedness.

Suppose that X is locally path-connected and connected. Consider the path-connected component $C \subset X$ of one of its points. Then C is open (by Theorem 9.22) and its complement $X - C$ is also open (as its the union of the other path-connected components). Since X is connected and $C \neq \emptyset$ then $C = X$, i.e. X is path-connected. □